Preliminary Exam, Algebra, January 6, 2010

1. (a) Let C be a finite cyclic group of order n. Show that $\mathrm{Aut}(G)$ is isomorphic to $(\mathbb{Z}/n\mathbb{Z})^\times$, the unit group of the ring $\mathbb{Z}/n\mathbb{Z}$.

(b) Let G be a group of order pq where $q < p$ are primes such that q does not divide $p-1$. Show that G is cyclic.

(c) Show that the symmetric group $\mathrm{Sym}(5)$ has no subgroup of order 15.

2. Let G be a finite group, let H be a normal subgroup of G, and let P be a Sylow p-subgroup of H for some prime p. Show that $G = HN_G(P)$.

3. Let $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$ be the field with 2 elements and let I be the ideal of $\mathbb{F}_2[X]$ generated by the element $f(X) = X^4 + X^3 + 1$. Show that that $\mathbb{F}_2[X]/I$ is a field.

4. Let K be a field and $T : K^3 \to K^3$ a K-linear transformation. Suppose that with respect to the standard basis $e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1)$, the matrix of T is

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(a) Find the matrix, call it B, that represents T with respect to the basis $(1,1,0), (0,1,1), (1,1,1)$.

(b) Show by explicit computation that A and B are similar matrices.

5. Let K be a field, V a finite-dimensional K-vector space, and $V^* = \mathrm{Hom}_K(V,K)$ the *dual space*.

(a) Given a basis $(v_1, \ldots, v_n)$ of V, define the *dual basis* of V^* and prove that it *is* a basis of the dual space.

(b) Prove that V and its double dual $(V^*)^*$ are isomorphic by an isomorphism that does not depend on any choice of basis.

6. Let K be a field, V a K-linear space, and $f\colon V \to V$ a K-linear transformation. Suppose that $v_i \in V$ is a family of *non-zero* vectors (I an index set) such that $f(v_i) = \lambda_i v_i$ with scalars $\lambda_i \in K$ such that $\lambda_i \neq \lambda_j$ whenever $i \neq j$. Prove that $\{v_i\}_{i \in I}$ is a *linearly independent* set.

7. Let R be a commutative ring.

(a) Prove that there is an R-module isomorphism $R/I \otimes_R R/J \cong R/(I+J)$.

(b) (Note: this is not really related to part (a).) Prove that there is an exact sequence of R-modules:

$$0 \to R/(I \cap J) \to R/I \oplus R/J \to R/(I+J) \to 0.$$

8. (a) Describe all the prime ideals of $\mathbb{C}[x]$.

(b) Describe all the prime ideals of $\mathbb{R}[x]$.

9. Find, with proof, all the subfields of $\mathbb{Q}(\sqrt[6]{5})$.