1. Let $G$ be a finite cyclic group of order $n$. Show that $\text{Aut}(G)$ is isomorphic to $(\mathbb{Z}/n\mathbb{Z})^*$, the unit group of the ring $\mathbb{Z}/n\mathbb{Z}$.

2. Let $G$ be a finite group, let $H$ be a normal subgroup of $G$, and let $P$ be a Sylow $p$-subgroup of $H$ for some prime $p$. Show that $G = HN_G(P)$.

3. Let $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$ be the field with 2 elements and let $I$ be the ideal of $\mathbb{F}_2[X]$ generated by the element $f(X) = X^4 + X^3 + 1$. Show that that $\mathbb{F}_2[X]/I$ is a field.

4. Let $K$ be a field and $T : K^3 \to K^3$ a $K$-linear transformation. Suppose that with respect to the standard basis $e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1)$, the matrix of $T$ is

$$A = \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}$$

(a) Find the matrix, call it $B$, that represents $T$ with respect to the basis $(1, 1, 0), (0, 1, 1), (1, 1, 1)$.

(b) Show by explicit computation that $A$ and $B$ are similar matrices.

5. Let $K$ be a field, $V$ a finite-dimensional $K$-vector space, and $V^* = \text{Hom}_K(V, K)$ the dual space.

(a) Given a basis $(v_1, \ldots, v_n)$ of $V$, define the dual basis of $V^*$ and prove that it is a basis of the dual space.

(b) Prove that $V$ and its double dual $(V^*)^*$ are isomorphic by an isomorphism that does not depend on any choice of basis.

6. Let $K$ be a field, $V$ a $K$-linear space, and $f : V \to V$ a $K$-linear transformation. Suppose that $v_i \in V$, $i \in I$, is a family of non-zero vectors ($I$ an index set) such that $f(v_i) = \lambda_i v_i$ with scalars $\lambda_i \in K$ such that $\lambda_i \neq \lambda_j$ whenever $i \neq j$. Prove that $\{v_i\}_{i \in I}$ is a linearly independent set.

7. Let $R$ be a commutative ring and let $I$ and $J$ be ideals of $R$.

(a) Prove that there is an $R$-module isomorphism $R/I \otimes_R R/J \cong R/(I + J)$.

(b) (Note: this is not really related to part (a).) Prove that there is an exact sequence of $R$-modules:

$$0 \to R/(I \cap J) \to R/I \oplus R/J \to R/(I + J) \to 0.$$  

8. (a) Describe all the prime ideals of $\mathbb{C}[x]$.

(b) Describe all the prime ideals of $\mathbb{R}[x]$.

9. Find, with proof, all the subfields of $\mathbb{Q}(\sqrt{5})$. 