Algebra Preliminary Exam Winter 2012

1. Recall that the commutator subgroup G' of a group G is defined as the subgroup of G generated by the set of elements of the form $xyx^{-1}y^{-1}$ with $x, y \in G$.

   (a) Show that if $f\colon G \to H$ is a group homomorphism then $f(G') \leq H'$.

   (b) Show that G' is a normal subgroup of G and that G/G' is abelian.

   (c) Show that for a subgroup H of G one has:

   $$G' \leq H \leq G \iff H \text{ is normal in } G \text{ and } G/H \text{ is abelian.}$$

2. Is the unit group of the ring $\mathbb{Z}/2012\mathbb{Z}$ cyclic? What is the largest possible order of an element of the unit group? Justify your answers.

3. Let $R := \mathbb{Z}[i] := \{a + bi \mid a, b \in \mathbb{Z}\}$ be the ring of Gaussian integers.

   (a) Show that $1 + i$ is a prime element in R.

   (b) Show that the ideal $I := (1 + i)R$ of R satisfies $I^2 = 2R$.

   (c) Show that $R/2R$ is a ring with 4 elements and that it is not isomorphic to $\mathbb{Z}/4\mathbb{Z}$, to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, or to the field with 4 elements.

4. Let L be a Galois extension of $\mathbb{Q}$ of degree 36. Show that there exists a Galois extension K of $\mathbb{Q}$ which is contained in L but different from $\mathbb{Q}$ and from L.

5. Suppose that V is a four-dimensional vector space over a field F. There exists a unique bilinear map $f : \bigwedge^2 V \times \bigwedge^2 V \to \bigwedge^4 V$ which satisfies

   $$f(x \wedge y, u \wedge v) = x \wedge y \wedge u \wedge v, \text{ for all } x, y, u, v \in V.$$

   Prove that f is a nondegenerate symmetric bilinear form.

   If $\{e_1, e_2, e_3, e_4\}$ is a basis of V, then the set $\{e_i \wedge e_j : 1 \leq i < j \leq 4\}$ is a basis of $\bigwedge^2 V$, and the set $\{e_1 \wedge e_2 \wedge e_3 \wedge e_4\}$ is a basis of $\bigwedge^4 V$.

   What is the matrix representing the bilinear form f, with respect to these bases?

6. A matrix X with entries in a field is called nilpotent if $X^n = 0$ for some positive integer n.

   (a) Prove that if a matrix is nilpotent, its determinant equals zero. Is the converse true? If not, give a counterexample.

   (b) Is the sum of two nilpotent matrices nilpotent? If not, give a counterexample.

   (c) Is the product of two nilpotent matrices nilpotent? If not, give a counterexample.

   (d) How many 2 by 2 nilpotent matrices are there with entries in $\mathbb{F}_q$?

7. Let p be a prime and let n be a positive integer. Consider the finite field $\mathbb{F}_{p^n}$ as a vector space over $\mathbb{F}_p$, and let A denote the linear transformation from $\mathbb{F}_{p^n}$ to itself defined by $a \mapsto a^p$ (the Frobenius map). Find the rational canonical form of A. (Hint: show that the minimal polynomial of A is $x^n - 1$ by proving that $A^n = 1$, but A satisfies no polynomial of smaller degree.)

8. Let R be a commutative ring, and let M and N be two projective R-modules. Prove that $M \otimes_R N$ is a projective R-module.

9. Find, with proof, all subfields of the splitting field of $x^4 - 2$ over $\mathbb{Q}$.