1. Recall that the commutator subgroup \( G' \) of a group \( G \) is defined as the subgroup of \( G \) generated by the set of elements of the form \( xyx^{-1}y^{-1} \) with \( x, y \in G \).

(a) Show that if \( f : G \rightarrow H \) is a group homomorphism then \( f(G') \leq H' \).

(b) Show that \( G' \) is a normal subgroup of \( G \) and that \( G/G' \) is abelian.

(c) Show that for a subgroup \( H \) of \( G \) one has:
\[
G' \leq H \leq G \iff H \text{ is normal in } G \text{ and } G/H \text{ is abelian.}
\]

2. Is the unit group of the ring \( \mathbb{Z}/2012\mathbb{Z} \) cyclic? What is the largest possible order of an element of the unit group? Justify your answers.

3. Let \( R := \mathbb{Z}[i] := \{a + bi \mid a, b \in \mathbb{Z}\} \) be the ring of Gaussian integers.

(a) Show that \( 1 + i \) is a prime element in \( R \).

(b) Show that the ideal \( I := (1 + i)R \) of \( R \) satisfies \( I^2 = 2R \).

(c) Show that \( R/2R \) is a ring with 4 elements and that it is not isomorphic to \( \mathbb{Z}/4\mathbb{Z} \), to \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \), or to the field with 4 elements.

4. Let \( L \) be a Galois extension of \( \mathbb{Q} \) of degree 36. Show that there exists a Galois extension \( K \) of \( \mathbb{Q} \) which is contained in \( L \) but different from \( \mathbb{Q} \) and from \( L \).

5. Suppose that \( V \) is a four-dimensional vector space over a field \( F \). There exists a unique bilinear map \( f : \Lambda^2 V \times \Lambda^2 V \rightarrow \Lambda^4 V \) which satisfies
\[
f(x \wedge y, u \wedge v) = x \wedge y \wedge u \wedge v, \text{ for all } x, y, u, v \in V.
\]
Prove that \( f \) is a nondegenerate symmetric bilinear form.

If \( \{e_1, e_2, e_3, e_4\} \) is a basis of \( V \), then the set \( \{e_i \wedge e_j : 1 \leq i < j \leq 4\} \) is a basis of \( \Lambda^2 V \), and the set \( \{e_1 \wedge e_2 \wedge e_3 \wedge e_4\} \) is a basis of \( \Lambda^4 V \).

What is the matrix representing the bilinear form \( f \), with respect to these bases?

6. A matrix \( X \) with entries in a field is called nilpotent if \( X^n = 0 \) for some positive integer \( n \).

(a) Prove that if a matrix is nilpotent, its determinant equals zero. Is the converse true? If not, give a counterexample.

(b) Is the sum of two nilpotent matrices nilpotent? If not, give a counterexample.

(c) Is the product of two nilpotent matrices nilpotent? If not, give a counterexample.

(d) How many 2 by 2 nilpotent matrices are there with entries in \( \mathbb{F}_q \)?
7. Let \( p \) be a prime and let \( n \) be a positive integer. Consider the finite field \( \mathbb{F}_{p^n} \) as a vector space over \( \mathbb{F}_p \), and let \( A \) denote the linear transformation from \( \mathbb{F}_{p^n} \) to itself defined by \( a \mapsto a^p \) (the Frobenius map). Find the rational canonical form of \( A \). (Hint: show that the minimal polynomial of \( A \) is \( x^n - 1 \) by proving that \( A^n = 1 \), but \( A \) satisfies no polynomial of smaller degree.)

8. Let \( R \) be a commutative ring, and let \( M \) and \( N \) be two projective \( R \)-modules. Prove that \( M \otimes_R N \) is a projective \( R \)-module.

9. Find, with proof, all subfields of the splitting field of \( x^4 - 2 \) over \( \mathbb{Q} \).