Algebra Preliminary Exam, Winter 2013

1. Show that $\mathrm{GL}_2(\mathbb{Z}/2\mathbb{Z})$ is isomorphic to $\mathrm{Sym}(3)$.

2. Assume that p is a prime and that P is a non-abelian group of order p^3. Show that $Z(P) = P'$ and that $|Z(P)| = p$.

3. Show that every finite integral domain is a field.

4. Let V be an n-dimensional vector space over a field F. Assume that T is a cyclic operator on V and that S is an operator on V which commutes with T. Prove that there is a polynomial $f(X) \in F[X]$ of degree at most $n-1$ such that $S = f(T)$.

5. Assume that S and T are diagonalizable operators on the n-dimensional vector space V. Prove that there exists an invertible operator Q such that $T = Q^{-1}SQ$ if and only if the characteristic polynomial of T is equal to the characteristic polynomial of S: $\chi_T(X) = \chi_S(X)$.

6. Let T be an operator on the 4-dimensional real vector space V and assume that the characteristic polynomial of T is $X^4 - 1$. Determine the minimial and characteristic polynomial of $\wedge^2(T)\colon \wedge^2(V) \to \wedge^2(V)$.

7. (a) Prove or disprove: $\mathbb{Z}/2\mathbb{Z}$ is a projective $\mathbb{Z}/4\mathbb{Z}$-module.

(b) Prove or disprove: $\mathbb{Z}/2\mathbb{Z}$ is a projective $\mathbb{Z}/6\mathbb{Z}$-module.

8. Suppose that M is an invertible $n \times n$ matrix with rational coefficients such that $M^{-1} = M^2 + M$. Prove that n is a multiple of 3. Furthermore show that, for fixed n, any two such M are similar.

9. Suppose that K/k is a finite Galois extension and that $\alpha_1, \ldots, \alpha_n$ are distinct elements of K. Suppose that $f(x) = (x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n)$ has coefficients in k. Prove that $f(x)$ is irreducible over k if and only if the action of $\mathrm{Gal}(K/k)$ on $\{\alpha_1, \ldots, \alpha_n\}$ is transitive. (Recall that the action is called transitive if for every i and j, there exists a $\sigma \in \mathrm{Gal}(K, k)$ such that $\sigma(\alpha_i) = \alpha_j$.)