1. Show that $\text{GL}_2(\mathbb{Z}/2\mathbb{Z})$ is isomorphic to $\text{Sym}(3)$.

2. Assume that $p$ is a prime and that $P$ is a non-abelian group of order $p^3$. Show that $Z(P) = P'$ and that $|Z(P)| = p$.

3. Show that every finite integral domain is a field.

4. Let $V$ be an $n$-dimensional vector space over a field $F$. Assume that $T$ is a cyclic operator on $V$ and that $S$ is an operator on $V$ which commutes with $T$. Prove that there is a polynomial $f(X) \in F[X]$ of degree at most $n - 1$ such that $S = f(T)$.

5. Assume that $S$ and $T$ are diagonalizable operators on the $n$-dimensional vector space $V$. Prove that there exists an invertible operator $Q$ such that $T = Q^{-1}SQ$ if and only if the characteristic polynomial of $T$ is equal to the characteristic polynomial of $S$: $\chi_T(X) = \chi_S(X)$.

6. Let $T$ be an operator on the 4-dimensional real vector space $V$ and assume that the characteristic polynomial of $T$ is $X^4 - 1$. Determine the minimal and characteristic polynomial of $\wedge^2(T) : \wedge^2(V) \rightarrow \wedge^2(V)$.

7. (a) Prove or disprove: $\mathbb{Z}/2\mathbb{Z}$ is a projective $\mathbb{Z}/4\mathbb{Z}$-module.
(b) Prove or disprove: $\mathbb{Z}/2\mathbb{Z}$ is a projective $\mathbb{Z}/6\mathbb{Z}$-module.

8. Suppose that $M$ is an invertible $n \times n$ matrix with rational coefficients such that $M^{-1} = M^2 + M$. Prove that $n$ is a multiple of 3. Furthermore show that, for fixed $n$, any two such $M$ are similar.

9. Suppose that $K/k$ is a finite Galois extension and that $\alpha_1, \ldots, \alpha_n$ are distinct elements of $K$. Suppose that $f(x) = (x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_n)$ has coefficients in $k$. Prove that $f(x)$ is irreducible over $k$ if and only if the action of $\text{Gal}(K/k)$ on $\{\alpha_1, \ldots, \alpha_n\}$ is transitive. (Recall that the action is called transitive if for every $i$ and $j$, there exists a $\sigma \in \text{Gal}(K,k)$ such that $\sigma(\alpha_i) = \alpha_j$.)