Algebra Preliminary Exam, Winter 2014

1. Let G be a finite group of odd order and let H be a subgroup of index 3. Show that H is normal in G. (Hint: Use the permutation representation of the action of G on the set of cosets G/H.)

2. Show that each Sylow-2-subgroup of $\mathrm{GL}_3(\mathbb{F}_2)$ is isomorphic to D_8. (Hint: Consider the subgroup of upper triangular matrices.)

3. Let R be a commutative ring and let S be a multiplicatively closed subset of R which contains 1_R. Moreover, let S' denote the set of those elements $s' \in R$ which divide some element of S (thus $S \subseteq S'$). Show that there exists a ring isomorphism $S^{-1}R \to S'^{-1}R$.

4. (a) Let V be a complex vector space and let $\langle -, - \rangle \colon V \times V \to \mathbb{C}$ be a function. Define what it means for $\langle -, - \rangle$ to be an *inner product* on V.

(b) Assuming $(V, \langle -, - \rangle)$ is an inner product space, define what is meant by a *unitary operator* on V.

(c) Let T be an operator on a finite dimensional complex vector space V and assume that the minimal polynomial of T is $\mu_T(x) = x^k - 1$ for some positive integer k. Prove that there exists an inner product $\langle -, - \rangle$ on V such that T is a unitary operator with respect to this product.

5. Let A be a real symmetric $n \times n$ matrix.

(a) Define what it means for A to be *positive definite*.

(b) Show that the following are equivalent: (i) A is positive definite; (ii) A is congruent to the identity matrix; (iii) there exists an invertible $n \times n$-matrix Q such that $A = Q^{\mathrm{tr}}Q$.

6. Assume that V and W are finite dimensional vector spaces over a field $\mathbb{F}$. Let $(v_1, \ldots, v_n)$ be a linearly independent sequence from V and let $w_1, \ldots, w_n$ be elements of W such that

$$\sum_{i=1}^{n} v_i \otimes w_i = 0_{V \otimes_{\mathbb{F}} W} .$$

Show that $w_1 = \cdots = w_n = 0_W$.

7. Let R be an integral domain and M a finitely generated and torsion-free R-module. Prove that M is isomorphic to a submodule of a finitely generated free R-module. Hint: Let F be a maximal finitely generated free submodule of M. (You need to prove that one exists; it may not be unique. Hint for this: take a maximal linearly independent subset of a finite generating set for M.) If $F \neq M$, show that there exists a nonzero $a \in R$ such that multiplication by a is an isomorphism from M to a submodule of F.

8. Let $K = \mathbb{F}_5(t)$, where t is an indeterminate. Let L be the splitting field of the polynomial $x^3 - t \in K[x]$. Describe all subfields of L containing K. How is the answer different if $\mathbb{F}_5$ is replaced by $\mathbb{F}_7$?

9. Let k be a field. Prove that $k[x]$ is not flat as a module over the ring $k[x^2, x^3]$.