Algebra Preliminary Exam, Winter 2015

1. Suppose that $K/\mathbb{Q}$ is a Galois extension of degree 70. Show that there exists a unique field extension $L/\mathbb{Q}$ of degree 14 such that $L \subseteq K$. Moreover show that $L/\mathbb{Q}$ is a Galois extension.

2. Compute the order of the element $32 + 2015\mathbb{Z}$ in the multiplicative group $(\mathbb{Z}/2015\mathbb{Z})^\times$. (Hint: Use the Chinese Remainder Theorem.)

3. Let G denote the symmetric group on 3 letters. Show that $\mathrm{Aut}(G)$ is isomorphic to G.

4. Assume that V is a finite dimensional vector space over the field $\mathbb{F}$ and that T is a linear operator on V with minimal polynomial $\mu_T(x) = p(x)^k$, where $k \geq 1$ and $p(x)$ is an irreducible polynomial in $\mathbb{F}[x]$ of degree d. Prove that the number of invariant factors of T is equal to $\frac{1}{d}\dim(\mathrm{Ker}(p(T))$.

5. Let $(V, \langle -, - \rangle_V)$ be a finite dimensional complex inner product space and let $T \colon V \to V$ be a linear operator. Prove that T is self-adjoint if, and only if, T is normal and every eigenvalue of T is real.

6. Let V_1 and V_2 be finite dimensional vector spaces over the field $\mathbb{F}$ and let $S_i \colon V_i \to V_i$ be linear operators, for $i = 1, 2$. Prove that $\mathrm{Trace}(S_1 \otimes S_2) = \mathrm{Trace}(S_1) \cdot \mathrm{Trace}(S_2)$.

7. Prove directly that if R is a Noetherian ring then R^n is a Noetherian R-module. (Hint: Let M be a submodule of R^n. Consider the set of first coordinates of elements of M; show this is a submodule of R and hence finitely generated. Use this to reduce to showing that $M \cap (0 \oplus R^{n-1})$ is finitely generated and use induction.)

8. Let $K = \overline{\mathbb{F}}_p(x, y)$ and $k = \overline{\mathbb{F}}_p(x^p, y^p)$. Show that $[K : k] = p^2$ and that there are infinitely many fields L satisfying $K \supset L \supset k$ and $[L : k] = p$.

9. Let k be a field, and let $A = k[x, y], B = k[u, v]$. View B as an A-module via the k-algebra homomorphism $f : A \to B$ defined by $f(x) = u$ and $f(y) = uv$. Prove that B is not flat as an A-module. (Hint: consider the maximal ideal $(x, y) \subset A$; show that the inclusion $(x, y) \subset A$ does not remain injective after tensoring with B over A by explicitly writing down an element in the kernel.)