Algebra Preliminary Exam
1/8/2016

1. If G is a nonabelian group of order p^3 for some prime p, prove that $Z(G)$ is equal to the commutator subgroup $[G, G]$.

2. For any finite group G and prime p, we let $n_p(G)$ denote the number of Sylow p-subgroups of G. Now suppose K is a normal subgroup of G, and let P be a Sylow p-subgroup of G.

   (a) Show that $n_p(G/K)$ divides $n_p(G)$.

   (b) Prove that $n_p(G/K) = n_p(G)$ if and only if P is a normal subgroup of PK.

3. Let R be a commutative ring with 1. Prove that $a \in R$ is nilpotent if and only if $a \in P$ for every prime ideal P of R.

4. Prove that every $\mathbf{Z}$-module is a submodule of an injective $\mathbf{Z}$-module.

5. Calculate with proof the Galois group of $x^{17} - 3$ over $\mathbf{Q}$. Your answer should be in terms of the semidirect product of two abelian groups.

6. Show that if A is a square matrix over a field such that $A^2 = A$, then A is similar to a diagonal matrix with only 0's and 1's on the diagonal.

7. Let A be a real 3×3 orthogonal matrix. Show that A has 1 or -1 as an eigenvalue.

8. Consider the set S of real 4×4 symmetric matrices, on which $G = GL_4(\mathbb{R})$ acts as:

   $$(s, g) \mapsto (g^t)sg.$$

   How many G-orbits are there in S?

9. Suppose that T is a linear operator on a 2-dimensional $\mathbb{F}_5$-vector space, and that the trace and the determinant of T are both equal to 1.

   (a) Prove or disprove: T is diagonalisable over $\mathbb{F}_5$.

   (b) Compute the trace of $\mathrm{Sym}^2 T$.