Algebra Prelim, Winter 2017

1. Show that for a group G the following are equivalent:
(i) G is abelian.
(ii) The map $G \to G$, $x \mapsto x^{-1}$ is an automorphism of G.
(iii) The subgroup $\Delta(G) := \{(x,x) \mid x \in G\}$ of $G \times G$ is normal in $G \times G$.
(iv) The map $G \times G \to G$, $(x,y) \mapsto xy$, is a homomorphism.

2. Let G be a group and let H be a subgroup of finite index n. Show that for every element $g \in Z(G)$ one has $g^n \in H$. Hint: Consider the subgroup $C := \langle g \rangle$ and show that the left multiplication action of C on G/H has orbits of size $|C/(C \cap H)|$.

3. Let R_1 and R_2 be rings and set $R = R_1 \times R_2$. Prove or disprove by a counterexample: Every ideal of R is of the form $I_1 \times I_2$ for an ideal I_1 of R_1 and an ideal I_2 of R_2.

4. Let A be an orthogonal 3×3 real matrix of determinant 1. Prove that there exists a nonzero vector $v \in \mathbb{R}^3$ fixed by A, i.e., $Av = v$.

5. Let V be the real vector space consisting of real 2×2 matrices of trace zero. Consider the symmetric bilinear form $\langle -, - \rangle$ on V given by the formula:

$$\langle A, B \rangle = \mathrm{Tr}(AB)$$

(Here AB is the 2×2 matrix that is the product of the 2×2 matrices A and B, and Tr denotes the trace.) Compute the signature of $\langle -, - \rangle$.

6. Prove, or disprove with a counterexample, the following statement: Any complex symmetric matrix is diagonalisable.

7. For each case, given an example of a commutative ring R and a module M over R with the described properties (you can use a different ring and module for each case):

- projective but not free.
- injective but not projective.
- torsion but not finitely generated.
- torsion-free and finitely generated but not free.

8. Calculate the Galois group of $x^4 - 2x^2 - 2$ over $\mathbb{Q}$.

9. Let A be an endomorphism of a finite dimensional vector space V over a field F. Show that V can be decomposed as $V = W \oplus W'$ where $A(W) \subseteq W$ and $A(W') \subseteq W'$ such that A acts invertibly on W and nilpotently on W'.