Algebra Preliminary Exam, Winter 2018

1. Show that the alternating group A_5 on 5 letters does not have a subgroup of order 20. (You can use that A_5 is not solvable, but not that A_5 is simple).

2. Let $T := \mathbb{F}_2[X]/(X^5 - 1)$, where $\mathbb{F}_2$ is the field with 2 elements.

(a) Show that the ring T is isomorphic to $\mathbb{F}_2 \times \mathbb{F}_{16}$, where $\mathbb{F}_{16}$ is a field with 16 elements.

(b) Compute all factor rings of T and compute the orders of their unit groups.

3. The goal of this Problem is to show that there exists no ring R with precisely 5 units. Suppose that R is a ring with unit group $R^\times = \langle u \rangle$ of order 5.

(a) Show that $1 + 1 = 0$ in R and that the subset $\mathbb{F}_2 := \{0, 1\}$ of R is a subring isomorphic to the field with 2 elements.

(b) Show that the subring $S := \mathbb{F}_2[u]$ of R, generated by u, is again a ring with unit group $S^\times = \langle u \rangle$ of order 5.

(c) Show that S is a factor ring of the ring $T := \mathbb{F}_2[X]/(X^5 - 1)$ and derive a contradiction using Problem 2(b).

4. Prove that any complex square matrix has a decomposition $A = B + C$ where B is diagonalizable and C is nilpotent such that $BC = CB$.

5. Assume that both A and B are real symmetric matrices of the same size, and A is positive definite. Prove that there is an invertible real matrix P such that $P^t A P = I_n$ is the identity matrix and $P^t B P$ is a diagonal matrix.

6. Prove that there is no 3×3 rational matrix A such that $A^8 = I_3$ but $A^4 \neq I_3$.

7. Let $L = \mathbb{R}(x, y)$ be the field generated over the real numbers by two indeterminates x and y, and let $K \subset L$ be the subfield $K = \mathbb{R}(x^2 + y^2, xy)$. Find all subfields of L containing K.

8. Let p be a prime and $n \geq 1$ an integer. The Frobenius endomorphism $\varphi \colon \mathbb{F}_{p^n} \to \mathbb{F}_{p^n}$, $x \mapsto x^p$, can be viewed as an $\mathbb{F}_p$-linear transformation of the $\mathbb{F}_p$-vector space $\mathbb{F}_{p^n}$.

(a) Prove that the minimal polynomial and characteristic polynomial of φ are both equal to $x^n - 1$.

(b) Find the Jordan canonical form of φ. Hint: your answer should involve the factorization $n = mp^t$ where p does not divide m.

9. If R is an integral domain with quotient field K, prove that $(K/R) \otimes_R (K/R) = 0$.