Algebra Preliminary Exam, Winter 2019

1. Let $p < q$ be primes, $n \geq 0$ an integer and G a group of order pq^n. Show that G is solvable.

2. Let G be a non-trivial finite group and p a prime. Suppose that every subgroup $H \neq G$ has index divisible by p. Prove that the center of G has order divisible by p.

3. Let $\mathfrak{p}$ and $\mathfrak{q}$ be nonzero prime ideals in a PID R. Is it possible to have a proper inclusion $\mathfrak{p} \subsetneq \mathfrak{q}$? Carefully justify your answer.

4. Let $V = \mathbb{R}^4$ be the standard 4-dimensional real vectorspace. Prove or disprove: For every element $\omega \in \bigwedge_{\mathbb{R}}^2 V$, there exist $v_1, v_2 \in V$ such that $\omega = v_1 \wedge v_2$.

5. Let N be the subgroup of the free abelian group $M = \mathbb{Z}^3$ generated by the following 3 elements:

$$x = \begin{bmatrix} 4 \\ 8 \\ -4 \end{bmatrix}, \quad y = \begin{bmatrix} 0 \\ -12 \\ 12 \end{bmatrix}, \quad \text{and} \quad z = \begin{bmatrix} 10 \\ 40 \\ 10 \end{bmatrix},$$

and let $Q := M/N$.

(a) Compute the rank of Q.

(b) Compute the invariant factors of Q.

6. Prove or disprove: Let V be a complex vectorspace, T a $\mathbb{C}$-linear automorphism of V, and W a $\mathbb{C}$-linear subspace of V stable under T. Then T restricts to an automorphism of W.

7. Let $\zeta_8 \in \mathbb{C}$ be a primitive 8-th root of unity. Determine the number of fields $\mathbb{Q} \subseteq F \subseteq \mathbb{Q}(2^{\frac{1}{4}}, \zeta_8)$, such that $F/\mathbb{Q}$ is Galois.

8. Let $f(x) = x^6 + \ldots + x + 1 \in \mathbb{Q}[x]$ and let $\zeta_7 \in \mathbb{C}$ be a primitive 7-th root of unity. Let $K = \mathbb{Q}(\zeta_7)$ and let G be the Galois group of $K/\mathbb{Q}$. Consider the ring $R = \mathbb{Z}[\zeta_7]$ and the ideal $I = (3)$ in R. Prove that the ideal I is maximal in R. (**Hint:** Consider the ring $\mathbb{Z}[x]/(3, f(x))$.) The group G acts on R and I, and thus G acts on the finite field $F = R/I$. Describe the explicit isomorphism between G and the Galois group of F over its prime subfield induced by the action of G on R/I.

9. Let $f(x) = x^4 + 5x + 5 \in \mathbb{Q}[x]$. The polynomial f has discriminant equal to $5 \cdot 55^2$. Prove that the Galois group of f over $\mathbb{Q}$ is not a subgroup of A_4.