Preliminary Examination Analysis 2013 May 31

1. Let $f : X_0 \to Y$ be a uniformly continuous function, where Y is a complete metric space and X_0 is a dense subset of a metric space X.

(1) Show that f has a unique continuous extension to a function $\hat{f} : X \to Y$.
(2) Show that $\hat{f}$ is also uniformly continuous.

2. Let X be a Hausdorff topological space.

(1) Show that a compact subset in Y is closed in Y.
(2) Give an example of a (T1)-space where a compact subset is not closed.

3. (1) Let f be a biholomorphic map of the unit disc in the complex plane to itself such that $f(0) = 0$. Show that

$$f(z) = cz$$

for some complex number $|c| = 1$.
(2) Show that the zeros of a holomorphic function are isolated.

4. (1) Show that the following improper integrals are equal

$$\int_0^\infty \frac{\sin^2 x}{x^2} dx = \int_0^\infty \frac{\sin x}{x} dx.$$

(2) Calculate $\int_0^\infty \frac{\sin^2 x}{x^2} dx$.

5. (1) Let E be a subset of $[0,1]$ of Lebesgue measure 1. Show that E is dense in $[0,1]$.

(2) Show that the Borel sets of $\mathbb{R}^n$ are precisely the members of the σ-algebra generated by the compact sets.

6. Let λ be the Lebesgue measure on $(0,\infty)$ and μ is a σ-finite Borel measure satisfying

- $\mu << \lambda$
- $\mu(B) = \alpha\mu(\alpha B)$

for any $\alpha \in (0,\infty)$ and Borel set $B \subset (0,\infty)$. Suppose that $\frac{d\mu}{d\lambda}$ is a continuous function. Show that

$$\frac{d\mu}{d\lambda}(x) = \frac{c}{x^2}$$

for some nonnegative number c and $x \in (0,\infty)$.

7. Suppose one wants to use the closed graph Theorem to prove a version of the open mapping Theorem. One may do it in two steps.

(1) Suppose that X and Y are Banach spaces and $T : X \to Y$ is a bijective bounded linear operator. Show that T^{-1} is also bounded.

(2) Suppose that X and Y are Banach spaces and $T : X \to Y$ is a surjective bounded linear operator. Show that T is open.

8. Recall that $I + T$ is a Fredholm operator with index zero if $T : X \to X$ is a compact operator of a Banach space X.

(1) Show that any non-zero spectrum of T has to be in the point spectrum.

(2) Show that

$$T(u)(t) = \int_0^t u(s)ds : C[0,1] \to C[0,1]$$

is a compact operator and find the spectrum $\sigma(T)$, where $C[0,1]$ is the space of all complex-valued continuous function on $[0,1]$ with

$$\|u(t)\| = \max_{t\in[0,1]} |u(t)|.$$