PRELIMINARY EXAMS IN ANALYSIS 2000 FALL

1. Suppose that X is a subset of the reals. Then X is said to be G_δ if it is an intersection of a countable number of open sets of the reals. Show that an intersection of a countable number of dense G_δ sets in the reals is again a dense G_δ set in the reals. (Hint: Baire Category Theorem)

2.

a. Suppose that X is a subset of the reals. Then X is said to be connected if there does not exist disjoint open sets U_1, U_2 such that $X = (X \bigcap U_1) \bigcup (X \bigcap U_2)$, where both $X \bigcap U_1$ and $X \bigcap U_2$ are nonempty. Show that a subset of the reals is connected if and only if it is an interval.

b. Suppose that X is a subset of the reals. Then X is said to be totally disconnected if, for any open interval (a, b), $X \bigcap (a, b)$ is not connected. Show that the Cantor set is a totally disconnected subset in the reals.

3. Let A be a measurable set in R^2. Prove that A has measure zero if and only if almost all sections $A_y = \{x : (x, y) \in A\}$ has measure zero.

4.

a. Give an example of a sequence of continuous functions on $(0, 1)$ which converge pointwise to a function, but do not converge in L_2.

b. Prove that if a sequence of continuous functions on $(0, 1)$ converge uniformly to a function, then they converge to that function in L_2.

5.

a. Compute

$$\frac{1}{2\pi i} \int_C \frac{dz}{sin(1/z)}$$

where C is the circle $|z| = 1/3$ with positive orientation.

b. Compute the Fourier transform of the Gaussian distribution $e^{-\frac{x^2}{2}}$, i.e.

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}} e^{ixy} dx$$

given

$$\int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}} = \sqrt{2\pi}.$$

6. Suppose that Ω is a bounded region in the complex plane C. And suppose that $\{f_n\}$ is a sequence of functions which are continuous on $\bar{\Omega}$ and holomorphic in Ω. Show that if $\{f_n\}$ converges uniformly on $\partial\Omega$, then $\{f_n\}$ uniformly converges on $\bar{\Omega}$.

7. Write

$$Tf(x) = \int_0^x f(s)ds.$$

a) Show that T defines a bounded linear operator on the Banach space $C[0,1]$, endowed with its usual norm.

b) Show that this operator on $C[0,1]$ is compact.

8.

a. Let H be a Hilbert space over the reals and let $T \in L(H)$ be a bounded linear operator on H. Prove that, if T is invertible that is $T^{-1} \in L(H)$, then so is $T^\star$ and

$$(T^\star)^{-1} = (T^{-1})^\star.$$

b. Let H be a Hilbert space over the reals and let $T \in L(H)$ be a bounded linear operator on H. Prove that both $T^\star T$ and $TT^\star$ are positive operators on H.