Analysis Prelim Fall 2001

Print Your Name:

All questions should be answered on this exam using the backs of sheets if necessary. The exam has 11 pages.

Show all your work and justify all claims!

Good luck!!

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1. (10 pts.) Show that if $f \in L^1(\mathbf{R})$ then $\int_{-\infty}^{\infty} |f(x+h) - f(x)|\, dx \to 0$ as $h \to 0$.

2. a. (8 pts.) Find the Fourier coefficients $\hat{f}(k)$ for the function $f(x) = x$ with respect to the exponential system $e^{2\pi ikx}$ $(k \in \mathbf{Z})$ on $[-\frac{1}{2}, \frac{1}{2}]$.

b. (8 pts.) Use the result of part a. to compute

$$\sum_{k=1}^{\infty} \frac{1}{k^2}.$$

3. (10 pts.) Show that if $f_n(x) = \cos nx$ then $f_n \to 0$ weakly in $L^2(-\pi, \pi)$ but $f_n \not\to 0$ in measure.

4. (10 pts.) Let X be a Banach space, Y a normed linear space. Suppose $T_n : X \to Y$ are linear and continuous, and $\{T_n x\}$ converges to Tx for all $x \in X$. Show that T is linear and continuous.

5. a. (8 pts.) Show that one can have strict inequality in Fatou's lemma.

b. (8 pts.) Show that the conclusion of the monotone convergence theorem can fail if the sequence $\{f_n\}$ is decreasing.

6. (10 pts.) Show that there does not exist a sequence of continuous functions $f_n : \mathbf{R} \to \mathbf{C}$ such that the sequence $\{f_n(x)\}$ is bounded if and only if x is irrational. (Hint: Show that the set $\{x : \{f_n(x)\} \text{ is bounded}\}$ is an F_σ.)

7. (10 pts.) Suppose $P_n(z) = 1 + z/1! + z^2/2! + \cdots + z^n/n!$. What can be said about the zeros of P_n as $n \to \infty$?

8. (10 pts.) Expand

$$\frac{1}{1-z^2}+\frac{1}{3-z}$$

in a series $\sum_{n=-\infty}^{\infty} c_n z^n$ which converges for $1 < |z| < 3$.

9. (8 pts.) Suppose $f(z)$ is analytic for $|z| < 1$ and satisfies $|f(z)| \leq 1$ there. Show that

$$|f'(0)| \leq 1 - |f(0)|^2.$$