Analysis Prelim
Fall 2002

1. Let (X, ρ) be a metric space with $\{x_n\}$ and $\{y_n\}$ two Cauchy sequences. Show that $\lim_{n\to\infty} \rho(x_n, y_n)$ exists.

2. Let (X_1, τ_1) and (X_2, τ_2) be topological spaces. Show that if $Y \subset X_1$ is connected and $f: X_1 \to X_2$ is continuous, that $f(Y)$ is connected.

3. Let $U \subset \mathbf{R}^n$ be connected and open. $f: U \to R$ is said to be Hölder continuous if for $\alpha > 0$

$$\sup_{x \neq y} \frac{|f(x) - f(y)|}{|x-y|^\alpha} < \infty$$

Show that if f is Hölder with $\alpha > 1$, that f is constant.

4. Is it possible to solve

$$xy^3 + xzu + yv^2 = 3$$

$$u^3yz + 2xv - u^2v^2 = 2$$

for C^1 mappings $u(x,y,z)$, $v(x,y,z)$ near $(x,y,z) = (1,1,1)$ and $(u,v) = (1,1)$?

5. Show that every closed and bounded set in $\mathbf{R}^n$ is compact.

6. Show (a) $\int_0^{2\pi} \sin m\theta \cos m\theta d\theta = 0$,
(b) Use (a) to show that bounded closed sets in $L_2[0,2\pi]$ need not be compact.

7. Suppose that $\{E_k\}_{k=1}^\infty$ is a sequence of Lebesgue measurable subsets of $[0,1]$, and that $m(E_k)=1$, $\forall$ k.

Show that $m(\bigcap_{k=1}^\infty E_k)=1$.

8. Suppose $f_k \varepsilon L^1(I) \cap L^\infty(I)$, $I = [a,b]$ with $f_k \to f$ in $L^1(I)$. Suppose that $\sup_k \|f_k\|_{L^\infty} < \infty$, prove that for $1<p<\infty$, $f \varepsilon L^1(I) \cap L^\infty(I)$ and $f_k \to f$ in $L^P(I)$.

9. One of the early triumphs of Cauchy was the evaluation of $\int_{-\infty}^{\infty} \frac{\sin x}{x} dx$, by means of the complex analysis he developed. Show that this integral equals π.