**Analysis Prelim, Fall 2008**

**1** (10 points). Let $S$ be a connected subset of the real numbers. Prove (using only the definition of connectedness) that if $a$ and $b$ are in $S$ and $a < c < b$ then also $c \in S$.

**2** (10 points). Consider the space $X = [0, 1] \times [0, 1] \times \cdots$ (the countably-infinite product of $[0, 1]$ with product topology). An element of $X$ may be thought of as a sequence $\{x_n\}_{n=1}^{\infty}$ with each $x_n \in [0, 1]$. Show that the function from $X$ to $\mathbb{R}$ defined by

$$\{x_n\} \mapsto \sum_{n=1}^{\infty} 2^{-n} x_n$$

is continuous.

**3**

(a) (10 points). Prove that the set of continuous compactly supported functions is dense in $L^p(\mathbb{R})$ for $1 \leq p < \infty$. (You may use Urysohn's lemma in the proof.)

(b) (10 points). Prove that every function $f \in L^p(\mathbb{R})$ is continuous in the mean, i.e., for any $\epsilon > 0$ there is a $\delta > 0$ such that for $|t| < \delta$

$$\int_{-\infty}^{\infty} |f(x+t) - f(x)|^p \, dx < \epsilon.$$

**4** (10 points). Construct a monotone function on $[0, 1]$ which is discontinuous at every rational point.

**5** (10 points). Let $T \colon C([0, 1]) \to C([0, 1])$ be defined by

$$Tf(x) = \int_0^x f(t) \, dt.$$

Prove that $T$ is a compact operator, i.e., the image of the unit ball is pre-compact.

**6** (15 points). Let $f$ be a compactly supported, infinitely differentiable function on $\mathbb{R}$ such that $\int_{-\infty}^{\infty} f(x) x^n \, dx = 0$ for all integers $n \geq 0$. Prove that $f \equiv 0$. (Hint: Pass to the Fourier transform of $f$.)

**7** (10 points). Evaluate the following integral:

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^4}.$$

**8** (15 points). Let $f_n$ be a sequence of holomorphic functions on a domain $D$. Show that if $f_n(z)$ converges uniformly on each compact subset of $D$, then so does $f_n'(z)$.