Algebra Preliminary Exam, Fall 2012

1. Show that all groups of order 36 are solvable.

2. Let H be a proper subgroup of a finite group G. Show that there exists an element $g \in G$ which is not conjugate to any element of H. (Hint: bound the number of elements in $\bigcup_{g \in G} gHg^{-1}$.)

3. Let $F := \boldsymbol{Z}/2\boldsymbol{Z}$.

(a) Show that $X^2 + X + 1$ is the only irreducible polynomial of degree 2 in $F[X]$.

(b) Show that $X^4 + X^3 + 1$ is irreducible in $F[X]$.

(c) Show that $X^4 + X^3 + 1$ is irreducible in $\mathbb{Q}[X]$.

4. Let $(V, \langle\ ,\ \rangle)$ be a finite dimensional complex inner product space. Recall that an operator T on V is said to be *unitary* if for every pair $(\boldsymbol{u}, \boldsymbol{v})$ of vectors from V, $\langle T(\boldsymbol{u}), T(\boldsymbol{v})\rangle = \langle \boldsymbol{u}, \boldsymbol{v}\rangle$. Prove that T is unitary if and only if the following statement holds: There exists an orthonormal basis $\mathcal{B} = (\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n)$ consisting of eigenvectors of T and every eigenvalue of T has norm 1.

5. Let T be a finite dimensional vector space and T an operator on V.

(a) Assume $V = \langle T, \boldsymbol{u}\rangle \oplus \langle T, \boldsymbol{w}\rangle$ where $\mu_{T,\boldsymbol{u}}(x) = \mu_{T,\boldsymbol{w}}(x) = p(x)$ is irreducible. Prove there is a T-invariant subspace X such that $U \cap X = W \cap X = \{\boldsymbol{0}\}$.

(b) Let $V = U \oplus W$ where U and W are T-invariant. Set $T_U = T|U$, $T_W = T|W$, $f(x) = \mu_{T_U}(x)$, $g(x) = \mu_{T_W}(x)$. Assume for every T-invariant subspace X that $X = (X \cap U) + (X \cap W)$. Prove that $f(x)$ and $g(x)$ are relatively prime.

6. Let $\mathrm{GL}_n(\mathbb{R})$ denote the group of $n \times n$ non-singular matrices with real coefficients and $\mathrm{Sym}_n(\mathbb{R})$ the space of $n \times n$ symmetric matrices. Define an action of $\mathrm{GL}_n(\mathbb{R})$ on $\mathrm{Sym}_n(\mathbb{R})$ by $Q \circ A = Q^{tr} A Q$. Determine, with a proof, the number of orbits for this action.

7. Let k be a field and let $A = k[x, y]$. Consider the maximal ideal $m = (x, y)$. Is m a projective A-module? Is m a flat A-module?

8. Let F be a field. Let $f(x) \in F[x]$ be the characteristic polynomial of some matrix with coefficients in F. Show that all matrices with coefficients in F with characteristic polynomial $f(x)$ are similar over F if and only if the factorization of $f(x)$ into irreducibles in $F[x]$ has no repeated roots.

9. Let p be an odd prime. Find, with proof, the Galois group of $x^p - 2$ over $\mathbb{Q}$. (Hint: your answer will be a semi-direct product of two cyclic groups.)