Preliminary Exam Analysis – Fall 2013

10/11/2013

Please state any theorem you use. Good Luck !!

1. Show that every closed subset of a compact topological space is compact.

2. Let $S \subseteq \mathbb{R}^2$ be the topologist's sine curve,
$$S = \left\{ x \times \sin(\tfrac{1}{x}) \,:\, x \in (0,1] \right\}.$$
Determine the closure $\overline{S}$ and show that $\overline{S}$ is connected.

3. Examine whether the following statements are true for a measure μ on a measurable space (X, Λ). Give a proof or a counter-example.
(a) Let $A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots$ be measurable sets. Then
$$\lim_{n\to\infty} \mu(A_n) = \mu(\bigcup_{n=1}^{\infty} A_n).$$
(b) Let $B_1 \supseteq B_2 \supseteq B_3 \supseteq \ldots$ be measurable sets. Then
$$\lim_{n\to\infty} \mu(B_n) = \mu(\bigcap_{n=1}^{\infty} B_n).$$
Note: We do not assume that μ is finite.

4. Show that for every nonnegative Lebesgue integrable function $f : [0,1] \to \mathbb{R}$ and every $\epsilon > 0$ there exists $\delta > 0$ such that for each measurable set $A \subseteq [0,1]$ with $m(A) < \delta$ it follows that $\int_A f(x)\,dx < \epsilon$.

5. Suppose H is a Hilbert space. Show that, if a sequence x_n weakly converges to x in H and $\|x_n\| \to \|x\|$, then x_n strongly converges to x in H.

6. Suppose X and Y are Banach spaces and $T : X \to Y$ is a bounded linear operator. Show that $R(T) = N(T^*)^\perp$ if the range $R(T)$ is closed. Recall that the annihilator space is as defined $M^\perp := \{\, y \in Y \,:\, \psi(y) = 0 \text{ for all } \psi \in M \,\}$, $M \subseteq X'$.

7. Prove the Casorati-Weierstrass Theorem: Let U be a neighborhood of an essential singularity c of a holomorphic f function on $U \setminus \{c\}$. Show that $f(U \setminus \{c\})$ is dense in $\mathbb{C}$.

8. Show that if $f(z)$ is an entire non-zero function, then $f(z) = e^{g(z)}$ for some entire function $g(z)$.