Preliminary Exam Analysis – Fall 2014

10/17/2014

Please state any theorem you use. Good Luck !!

1. Suppose that X and Y are topological spaces and $X \times Y$ is the product space. Show that $A \times B \subset X \times Y$ is closed if A and B are closed in X and Y, respectively.

2. Suppose $\{X_\alpha\}$ is a collection of topological spaces and Y is another topological space. Show that a map $f : Y \to X$ to the product space $X = \prod_\alpha X_\alpha$ (with product topology) is continuous if and only if all the maps $\pi_\alpha \circ f : Y \to X_\alpha$ are continuous. (Here $\pi_\alpha : X \to X_\alpha$ are the projection maps.)

3. (a) State Fatou's lemma.
   (b) Show that the conlcusions of Fatou's lemma may fail if the functions are not necessarily nonnegative.
   (c) Show that the conclusion still holds, if we assume there is an integrable function g such that $|f_n| \leq g$.

4. Prove or disprove:
   (a) $L^2([0,1])$ convergence implies convergence almost everywhere.
   (b) If f_n are measurable functions on $[0,\infty)$ which converge uniformly to zero as $n \to \infty$, then
$$\lim_{n\to\infty} \int_0^\infty f_n \, dx = 0.$$

5. Let X be a Banach space with two closed subspaces X_0 and X_1 such that $X = X_0 \dot{+} X_1$ (i.e., every $x \in X$ is the unique sum $x = x_0 + x_1$ with $x_i \in X_i$). Show that the projection map P defined by $P(x_0 + x_1) = x_0$ is a bounded linear operator.

6. Let X be Banach space, and let X_0 be a one-dimensional subspace of X. Show that there exists a closed subspace X_1 of X such that $X = X_0 \dot{+} X_1$.

7. Let f be analytic on an open neighborhood of the annulus $A = \{ z \in \mathbb{C} : 1 \leq |z| \leq 2 \}$. Assume that $|f(z)| \leq 1$ whenever $|z| = 1$ and $|f(z)| \leq 4$ whenever $|z| = 2$. Show that $|f(z)| \leq |z|^2$ for $z \in A$.

8. Show that there exists an analytic function $g(z)$ on $\Omega = \mathbb{C} \setminus [0,2]$ (the complex plane minus the interval $[0,2] \subset \mathbb{R}$) such that $g(z)^3 = z(z-1)(z-2)$.