Preliminary Exam Analysis – Fall 2015

10/9/2015

Please state any theorem you use. Good Luck !!

1. Show that a continuous function $f : [a, b] \to \mathbb{R}$ is always uniformly continuous.

2. Suppose that K is compact and U is open in the Euclidean space $\mathbb{R}^n$. Also assume that $K \subset U$. Show that there is an open subset V such that the closure $\overline{V}$ is compact and
$$K \subset V \subset \overline{V} \subset U.$$

3. (a) Show that if $|f_n| \leq g \in L^1(\Omega)$, then f_n are uniformly integrable in Ω.

   (b) Does there exist a uniformly integrable family of functions $\{f_n\}$ with no integrable function g such that $|f_n| \leq g$?

   (c) Let f_k and g_k be μ-measurable such that $f_k \to f$ and $g_k \to g$ μ-a.e., $|f_k| \leq g_k$ and $\int g_k \to \int g < \infty$. Show that $\int f_k \to \int f$.

4. Using the dominated convergence theorem or otherwise, prove that
$$\lim_{n\to\infty} \int_0^1 e^{1/x}(1 + n^2 x)^{-1} \sin(ne^{-1/x})\, dx = 0.$$

5. Let X be a Banach space and let τ be the canonical map from X into its second dual X^{**}. Give the definition of τ and show that τ is an isometry.

6. Let X be a Banach space. Show that a linear functional on X is continuous if and only if its kernel (nullspace) is closed. Does this statement still hold if X is a normed space (but not a Banach space) ?

7. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with the radius of convergence of this power series being $R > 0$.

   (a) Let $|z_0| < R$. Show directly that $f(z)$ can be expanded into a power series about z_0 with a radius of convergence $\geq r := R - |z_0| > 0$.

   (b) Show directly that $f(z)$ is differentiable for $|z| < R$ and that $f'(z) = \sum_{n=0}^{\infty}(n + 1)a_{n+1}z^n$ with this power series having radius of convergence R.

8. Let $\Omega \subset \mathbb{C}$ be a connected domain and let $f(z)$ be holomorphic on Ω. Show that $\mathrm{Re}[f(z)]$ cannot attain a maximum on Ω unless f is constant.