Analysis Preliminary Exam, Math @ UCSC, Fall 2016

1. Let $C[0,1] = \{f : [0,1] \to \mathbf{R} : \ f \text{ is continuous on } [0,1]\}$ with the norm
$$\|f\|_{C[0,1]} = \max\{|f(x)| : x \in [0,1]\}.$$
Similarly, let $C^1[0,1] = \{f : [0,1] \to \mathbf{R} : \ f \text{ and } f' \text{ are all continuous on } [0,1]\}$ with the norm
$$\|f\|_{C^1[0,1]} = \max\{|f(x)| + |f'(x)| : x \in [0,1]\}.$$
Show that a bounded subset in $C^1[0,1]$ is pre-compact in $C[0,1]$.

2. Show that there is no continuous function from $[0,1]$ into the Cantor set, except the constant functions.

3. (a) Let f_n be a sequence of measurable functions that converges in measure to f. Prove that there exists a subsequence f_{n_k} which converges almost everywhere.
(b) Give an example of a sequence f_n of measurable functions converging in measure, which do not converge almost everywhere.

4. (a) Let f be an integrable function on $[a,b]$. Show that the function $F(x) = \int_a^x f(t)\,dt$ is absolutely continuous.
(b) Let g be absolutely continuous and monotonically increasing. If E is a set of measure zero, then $g(E)$ has also measure zero.

5. Let V be a normed vector space. Prove that if $x_0 \in V$ and $x_0 \neq 0$, then there exists a continuous linear functional $\phi \in V^*$ such that
(a) $\phi(x_0) = \|x_0\|$; (b) $\|\phi\| = 1$.
Moreover, prove that
$$\|x_0\| = \sup_{\phi \in V^*,\, \|\phi\|=1} |\phi(x_0)|.$$

6. Show that $L^2[0,2]$ is a set of first category in $L^1[0,2]$.

7. Let $f_n(z)$ be a sequence of functions holomorphic in the connected open set Ω of the complex plane $\mathbf{C}$ and assume they converge uniformly on every compact subset of Ω. Show that the sequence of derivatives $f_n'(z)$ also converges uniformly on every compact subset of Ω.

8. Let $f(z)$ be holomorphic in $|z| \leq R$ with $|f(z)| \leq M$ on $|z| = R$ for some $M > 0$. Show that
$$|f(z) - f(0)| \leq \frac{2M|z|}{R}\,.$$
Moreover, use this to give a proof of Liouville's theorem.