1. Let $C[0, 1] = \{ f : [0, 1] \to \mathbb{R} : f \text{ is continuous on } [0, 1] \}$ with the norm

$$\|f\|_C[0, 1] = \max\{ |f(x)| : x \in [0, 1] \}.$$ 

Similarly, let $C^1[0, 1] = \{ f : [0, 1] \to \mathbb{R} : f \text{ and } f' \text{ are all continuous on } [0, 1] \}$ with the norm

$$\|f\|_{C^1[0, 1]} = \max\{ |f(x)| + |f'(x)| : x \in [0, 1] \}.$$ 

Show that a bounded subset in $C^1[0, 1]$ is pre-compact in $C[0, 1]$.

2. Show that there is no continuous function from $[0, 1]$ into the Cantor set, except the constant functions.

3. (a) Let $f_n$ be a sequence of measurable functions that converges in measure to $f$. Prove that there exists a subsequence $f_{n_k}$ which converges almost everywhere.

(b) Give an example of a sequence $f_n$ of measurable functions converging in measure, which do not converge almost everywhere.

4. (a) Let $f$ be an integrable function on $[a, b]$. Show that the function $F(x) = \int_a^x f(t) \, dt$ is absolutely continuous.

(b) Let $g$ be absolutely continuous and monotonically increasing. If $E$ is a set of measure zero, then $g(E)$ has also measure zero.

5. Let $V$ be a normed vector space. Prove that if $x_0 \in V$ and $x_0 \neq 0$, then there exists a continuous linear functional $\phi \in V^*$ such that

(a) $\phi(x_0) = \|x_0\|$; (b) $\|\phi\| = 1$.

Moreover, prove that

$$\|x_0\| = \sup_{\phi \in V^* : \|\phi\| = 1} |\phi(x_0)|.$$ 

6. Show that $L^2[0, 2]$ is a set of first category in $L^1[0, 2]$.

7. Let $f_n(z)$ be a sequence of functions holomorphic in the connected open set $\Omega$ of the complex plane $\mathbb{C}$ and assume they converge uniformly on every compact subset of $\Omega$. Show that the sequence of derivatives $f'_n(z)$ also converges uniformly on every compact subset of $\Omega$.

8. Let $f(z)$ be holomorphic in $|z| \leq R$ with $|f(z)| \leq M$ on $|z| = R$ for some $M > 0$. Show that

$$|f(z) - f(0)| \leq \frac{2M|z|}{R}.$$ 

Moreover, use this to give a proof of Liouville’s theorem.