Fall 2017 - Analysis Prelim

1. Prove that every compact Hausdorff space X is normal, i.e., for each pair A, B of disjoint closed subsets of X there exist disjoint open sets U, V such that $U \supseteq A$ and $V \supseteq B$.

2. Let X and Y be topoplogical spaces and let $X \times Y$ be the product space. Show that $A \times B \subseteq X \times Y$ is closed if A and B are closed in X and Y, respectively.

3. Suppose that f is a μ-a.e. finite measurable function on a finite measure space (X, Λ, μ). Show that, for any $\epsilon > 0$, there exists $M > 0$ such that $|f(x)| \leq M$ except on a set of measure less than ϵ.

4. Suppose that $f_n(x)$ is a monotone increasing sequence of μ-a.e. finite measurable functions on a finite complete measure space (X, Λ, μ). Then f_n pointwisely converges to a measurable function f μ-a.e. in X if it converges to f in measure.

5. Let $\{\alpha_n\}$ be a sequence of reals, and assume that the series

$$\sum_{n=1}^{\infty} \alpha_n x_n$$

converges for every $\{x_n\} \in l^2$. Show that $\{\alpha_n\} \in l^2$. l^2 is the Hilbert space of sequence $\{x_n\}$ of reals such that $\sum_n x_n^2 < \infty$.

6. Let $X := C([0,1], \mathbf{C})$ be the space of continuous complex-valued functions equipped with the super norm

$$\|f\|_\infty = \sup_{t \in [0,1]} |f(t)|.$$

Define $T : X \to X$ by

$$(Tf)(t) := tf(t), \ \forall f \in X, \ \forall t \in [0,1].$$

Determine the spectrum $\sigma(T)$, the point spectrum $\sigma_p(T)$, the residual spectrum $\sigma_r(T)$ and the continuous spectrum $\sigma_c(T)$ of T.

7. Compute

$$\int_0^{2\pi} \frac{\cos\theta}{4 + \cos\theta} \, d\theta,$$

by expressing this integral as a complex integral over the unit cirle and using residue theory.

8. How many roots does $3z^7 - 5z^3 + 1 = 0$ have in the annulus $\{1 < |z| < 2\}$?