Fall 2018 - Analysis prelim - Friday, October 5
University of California Santa Cruz

1. Let $\{x_n\}$ be a sequence of real numbers. Set
$$y_n = \frac{x_1 + ... + x_n}{n}.$$
Show that if $\lim_{n\to\infty} x_n = a \in \mathbb{R}$, then $\lim_{n\to\infty} y_n$ exists and please find $\lim_{n\to\infty} y_n$. What about the converse, that is does the existence of $\lim_{n\to\infty} y_n$ guarantee the existence of $\lim_{n\to\infty} x_n$? Justify your answer.

2. Show that the mapping $T : \mathbb{R} \to \mathbb{R}$ defined by
$$T(x) = \frac{\pi}{2} + x - \arctan(x)$$
has no fixed point in $\mathbb{R}$ and that
$$|T(x) - T(y)| \leq |x - y|$$
for all distinct $x, y \in \mathbb{R}$. Please explain why does this example not contradict the Contraction Mapping Theorem.

3. Let $A \subset \mathbb{R}$ be a measurable set with $m(A) < \infty$. (m: Lebesgue measure)
   (a) Show that the function $\varphi : \mathbb{R} \to [0, \infty)$ defined by $\varphi(x) := m(A \cap (-\infty, x])$ is continuous.
   (b) Prove that there exists $x \in \mathbb{R}$ such that $m(A \cap (-\infty, x)) = m(A \cap (x, \infty))$.

4. Let $f_n \colon [0, 1] \to \mathbb{R}$ be Lebesgue measurable functions, converging a.e. to a real-valued function f. Show that f is Lebesgue measurable.

5. Let V be a normed space. Give the definition of when V is called reflexive. Show that if V is a Banach space and if V^* is reflexive, then V is reflexive.

6. Let $A : X \to Y$ be a bounded linear operator (where X and Y are Banach spaces). Define the Banach space adjoint A^*, show that it is a well-defined bounded linear operator and that $\|A\| = \|A^*\|$.

7. For $0 < p < 1$, evaluate
$$\int_{-\infty}^{\infty} \frac{e^{px}}{1 + e^x}\, dx$$
by residues. Justify any limiting process carefully. [Hint: use a family of rectangles of vertical width 2π as contours of integration]

8. Let $m \geq 0$. How many roots does the function $f(z) = \sum_{k=0}^{m} \frac{z^k}{k!} + 3z$ have inside the unit disk? Justify your answer.