SPRING 2000

Mathematics Analysis Preliminary Exam
June 9th, 2000

1. Prove that every compact Hausdorff space is regular. ("Regular" means a closed set and a point not in that set can be separated by open sets.)

2. Let μ be a σ-finite positive measure on Borel sets in $\mathbb{R}$. Suppose that

$$L^1(\mathbb{R}, \mu) \subset L^\infty(\mathbb{R}, \mu).$$

Show that there exisits a constant $c > 0$ such that if A is a Borel set with $\mu(A) > 0$ then automatically $\mu(A) \geq c$.

3. Prove that

$$\mu(\{x : |f(x)| > s\}) \leq \frac{1}{s^p} \int_X |f|^p \, d\mu,$$

where (X, Λ, μ) is a measure space and $p > 0$.

4. Let $\lambda > 1$. Show that the equation

$$\lambda - z - e^{-z} = 0$$

has exactly one solution in the right half plane $\{z | Re(z) > 0\}$.

5. Let $u : \mathbb{C} \to \mathbb{R}$ be a harmonic function such that $u(z) \geq 0$ for all $z \in \mathbb{C}$. Prove that $u \equiv \text{const}$.

6. Let E and F be Hilbert spaces and let $A : E \to F$ be a linear operator that takes every convergent sequence to a weakly convergent sequence. Prove that A is bounded.

Hint: Show that for an unbounded operator A, there exists a sequence $f_n \to 0$ such that $\| Af_n \| \to \infty$. Then use the Banach-Steinhaus theorem to arrive to a contradiction.

7. Let f be an entire function which is also L^2. Prove that $f \equiv 0$.
Hint: Integrate $|f|^2$ over the disc of radius R in polar coordinates and express the result in terms of the Taylor expansion of f.