Name.

Analysis 'Preliminary' Exam. June, 2002.

Instructions. Choose a four digit number to identify yourself. Copy each problem you are planning to do on a separate sheet of paper. Make sure you put your identification number on each sheet you hand in. We do give partial credit for partial solutions, or ideas that are on the right track.

1. A conformal map of a single complex variable is a holomorphic map which is one-to-one. ~~(or whose derivative never vanishes)~~ Show that any conformal map taking the Riemann sphere (i.e. the extended complex plane) onto itself must be linear fractional, i.e. of the form $(az+b)/(cz+d)$.

2. Show that an analytic function $f(z)$ whose modulus $|f(z)|$ is everywhere constant must itself be a constant function.

3. Evaluate $\int_0^\infty x^2 dx/(x^2+a^2)^2$ using the method of residues. Here a is a fixed real number.

4. A. Define the Fourier transform of a Borel measure on the real line, assuming the measure of the whole real line is finite.

B. Compute the Fourier transform of $\delta_0 + \delta_1 + \ldots + \delta_N$ where δ_i is the Dirac mass concentrated at the point δ_i

C. Let μ be a Borel measure and $\phi(\xi) = \hat{\mu}(\xi)$ its Fourier transform, a function of $\xi \in \mathbb{R}$. Let N distinct points $\xi_1, \ldots \xi_N$ on the real line be given and form the expression $\Sigma \phi(\xi_i - \xi_j) z_i \bar{z}_j$. Prove that this expression, viewed as a function of the z_i, defines a positive definite quadratic form on $\mathbb{C}^N$.

5. T is a self-adjoint operator on a separable Hilbert space satisfying $T^2 = T$. T is neither 0 nor the identity I. Find the spectrum of T, proving the validity of what you find.

6. Let C denote the Cantor set.

A. Does there exist another topological space X such that $C \times X$ is connected? If so, construct X. If not, prove not.

B. Construct a continuous map from C onto the unit circle.

7. A map between topological spaces is called "open" if the image of any open set is open. Suppose that $f : X \to Y$ is a continuous open map, with X is compact and Y connected. Show that f is onto.

8. Write $\hat{f} = F(f)$ for the Fourier transform of a function f and write F^{-1} for the inverse Fourier transform. Fix a function p such that its Fourier transform $\hat{p} = Fp$ lies in $L_1(\mathbb{R}) \cap L_2(\mathbb{R})$. Define the operator A by

$$A(f) = F^{-1}(\hat{p}\hat{f}).$$

Here $\hat{p}\hat{f}$ denotes the pointwise multiplication of $\hat{f}$ and $\hat{p}$. Prove that A defines a bounded linear operator from $L_2(\mathbb{R})$ to $C_0(\mathbb{R})$, the latter being endowed with the standard (uniform) norm.