Algebra Preliminary Exam

June 10, 2005

1. (a) Define a composition series for a finite group G.

(b) Prove that if $n \geq 5$ then the symmetric group S_n has a unique composition series. (You may assume known that $Z(S_n) = 1$ if $n \geq 5$.)

2. Let G be a finite group and p a prime number. Suppose that P is a p-sylow subgroup of G and N is a normal subgroup of G. Prove the following

(a) $P \cap N$ is a p-Sylow of N.

(b) PN/N is a p-Sylow of G/N.

3. Let R be a domain and D a multiplicative subset of R (not containing zero element).

(a) Define the localization $D^{-1}R$.

(b) If R is a PID, prove that $D^{-1}R$ is also a PID.

4. (a) Find all possible Jordan canonical forms for a matrix whose characteristic polynomial is $(x-3)^2(x+4)^3$. Indicate which of these forms is the matrix of a cyclic transformation.

(b) Let V be a complex vector space of dimension 6 and let τ be a linear operator on V which has characteristic polynomial $(x-\alpha)^6$. If the rank of the operator $\tau - \alpha I_V$ is two what are the possible Jordan forms for τ?

5. (a) Let V be a finite dimensional vector space over a field $\mathbb{F}$ and let $\sigma : V \to V$ be a linear transformation. Define what it means for σ to be diagonalizable.

(b) Let V and W be a finite dimensional vector spaces over a field $\mathbb{F}$ and assume that $\sigma : V \to V$ and $\tau : W \to W$ are diagonalizable linear tranformations. Prove that $\sigma \otimes \tau : V \otimes_{\mathbb{F}} W \to V \otimes W$ is diagonalizable. Furthermore, prove if the eigenvalues of σ are $\alpha_1, \alpha_2, \ldots, \alpha_s$ and the eigenvalues of $\sigma \otimes_{\mathbb{F}} \tau$ are $\alpha_i\beta_j, 1 \leq i \leq s, 1 \leq j \leq t$.

6. Let V be a finite dimensional vector space over the field $\mathbb{F}$.

(a) Let $B : V \times V \to$ be a function. Define what it means for B to be a non-degenerate alternating form.

(b) Assume that $B : V \times V \to \mathbb{F}$ is an non-degenerate alternating form and let $\overline{v}$ be a non-zero vector from B and $c \in \mathbb{F}$. Define $\tau_{\overline{v},c} : V \to V$ by

$$\tau_{\overline{v},c}(\overline{w}) = \overline{w} - cB(\overline{w}, \overline{v})\overline{v}$$

Prove that $\tau_{\overline{v},c}$ is an isometry of (V, B) which fixes every vector in $\overline{v}^{\perp}$.

7. Let M be a finitely generated module over an integral domain.

(a) Prove that $x_1, ..., x_n \in M$ are linearly independent if and only if $\bar{x}_1 = x_1 + \mathrm{Tor}(M), ..., \bar{x}_n = x_n + \mathrm{Tor}(M) \in M/\mathrm{Tor}(M)$ are linearly independent.

(b) Prove or disprove that M and $M/\mathrm{Tor}(M)$ have the same rank.

8. Let n be a positive integer.

(a) Prove that $\mathbb{Q}(\xi_n)$ is a Galois extension of $\mathbb{Q}$ and determine the corresponding Galois group where $\xi_n = e^{2\pi i/n}$.

(b) Prove that $2^{1/5}$ is not contained in $\mathbb{Q}(\xi_n)$ for any n.

9. Determine the Galois group of $f(x) = x^5 - 6x + 3$ over $\mathbb{Q}$.