ANALYSIS PRELIMINARY EXAMINATION JUNE 2006

1. Show that a compact subset of a metric space is bounded.

2. Consider the space $X = [0,\ 1] \times [0,\ 1] \times \cdots$ (the countably-infinite product of $[0,\ 1]$ with itself with product topology). An element of X may be thought of as a sequence $\{x_n\}_{n=1}^{\infty}$ with each $x_n \in [0,\ 1]$. Is the function from X to R defined by

$$\{x_n\} \to \sum_{n=1}^{\infty} \frac{x_n}{n^2}$$

continuous?

3. Compute

$$\int_0^{\frac{\pi}{2}} \frac{1}{a + \sin^2 x} dx \qquad \text{where } a > 0.$$

4. Let $s \geq 0$. Describe the set of all entire functions $f(z)$ that satisfy

$$\sup_{z \in C} \frac{|f(z)|}{(1 + |z|)^s} < \infty$$

and prove your statement.

5. Let $(E,\ \mu)$ be a measure space.

(1) If E_1 and E_2 are measurable, show that

$$m(E_1 \bigcup E_2) + m(E_1 \bigcap E_2) = m(E_1) + m(E_2).$$

(2) Let g be an integrable function on E. And suppose $\{f_n\}$ is a sequence of measurable functions such that $|f_n| \leq g, \forall n = 1, 2, \dots$ a.e. on E. Show that

$$\int_E \underline{\lim}_n f_n d\mu \leq \underline{\lim}_n \int_E f_n d\mu \leq \overline{\lim}_n \int_E f_n d\mu \leq \int_E \overline{\lim}_n f_n d\mu.$$

6. Let f be a integrable function and g be a bounded measurable function. Then show that

$$\lim_{t \to 0} \int_R g(x)(f(x+t) - f(x)) dm(x) = 0$$

where m is the Lebesgue measure and R is the real line.

7. Let X be a Banach space.

(1) Show that a linear functional is continuous if and only if its kernel is closed.

(2) Let X_0 be a subspace of X. Show that X_0 is not dense in X if and only if

$$\exists f \in X^*, \quad f \neq 0 \quad \text{and} \quad f|_{X_0} = 0.$$

8. Let X be a Banach space and $T \in L(X)$ is a linear operator with $||T|| < 1$. Show that $I - T$ is invertible, where $I : X \to X$ is the identity operator.