1 (5 points). Show that the set of rational numbers is not the intersection of countably many open subsets of the set of real numbers with its usual topology.

2 (10 points). Prove that if X and Y are compact metric spaces then so is X × Y.

3 (10 points). Let (X, μ) be a measure space with μ(X) < ∞. Prove that L^p(X, μ) ⊂ L^q(X, μ) whenever 1 ≤ q < p ≤ ∞ and that the inclusion is continuous.

4 (a) (10 points). Let f : [0, 1] → R be an absolutely continuous function and let E ⊂ [0, 1] be a Lebesgue measurable set. Prove that f(E) is Lebesgue measurable.

(b) (10 points). Does the assertion of part (a) remain correct if f is just a continuous function? Give a proof or construct a counterexample.

5 Let T : C([0, 1]) → C([0, 1]) be defined by

\[ Tf(x) = \int_0^x f(t) \, dt. \]

(a) (5 points). Prove that T is a bounded operator and find \(\|T\|\).

(b) (10 points). Find the spectral radius and the spectrum of T.

6 Recall that \(l_\infty(\mathbb{C})\) is the Banach space formed by bounded sequences \(\{a_n\} = (a_1, a_2, \ldots)\) of complex numbers, equipped with the norm \(\|\{a_n\}\|_\infty = \sup_n |a_n|\), and that \(l_1(\mathbb{C})\) is the Banach space of sequences \(\{a_n\}\) of complex numbers such that the sum \(\sum_n |a_n|\) converges, equipped with the norm \(\|\{a_n\}\|_1 = \sum_n |a_n|\).

(a) (10 points). Prove that \(l_1(\mathbb{C})\) is separable, but that \(l_\infty(\mathbb{C})\) is not.

(b) (5 points). Prove that \(l_1(\mathbb{C})^* = l_\infty(\mathbb{C})\).

(c) (10 points). Prove that \(l_\infty(\mathbb{C})^* \neq l_1(\mathbb{C})\).

7 (5 points). Evaluate

\[ \int \cot^2(\pi z) \, dz \]

over the circle with center 0 and radius 1/2, described once counterclockwise.

8 (10 points). Let \(H = \{x ∈ \mathbb{C} : \Re(z) > 0\}\) and let \(\tilde{H} = H ∪ \mathbb{R}\) be its closure. Let f be a bounded and continuous function in \(\tilde{H}\), which is holomorphic in H and such that f(z) is real for all z ∈ R. Prove that f is constant.