**Analysis Prelim, Spring 2008**

**1** (5 points). Show that the set of rational numbers is not the intersection of countably many open subsets of the set of real numbers with its usual topology.

**2** (10 points). Prove that if $X$ and $Y$ are compact metric spaces then so is $X \times Y$.

**3** (10 points). Let $(X, \mu)$ be a measure space with $\mu(X) < \infty$. Prove that $L^p(X, \mu) \subset L^q(X, \mu)$ whenever $1 \leq q < p \leq \infty$ and that the inclusion is continuous.

**4**

- **(a)** (10 points). Let $f\colon [0, 1] \to \mathbb{R}$ be an absolutely continuous function and let $E \subset [0, 1]$ be a Lebesgue measurable set. Prove that $f(E)$ is Lebesgue measurable.
- **(b)** (10 points). Does the assertion of part (a) remain correct if $f$ is just a continuous function? Give a proof or construct a counterexample.

**5** Let $T\colon C([0, 1]) \to C([0, 1])$ be defined by

$$Tf(x) = \int_0^x f(t)\,dt.$$

- **(a)** (5 points). Prove that $T$ is a bounded operator and find $\|T\|$.
- **(b)** (10 points). Find the spectral radius and the spectrum of $T$.

**6** Recall that $l_\infty(\mathbb{C})$ is the Banach space formed by bounded sequences $\{a_n\} = (a_1, a_2, \dots)$ of complex numbers, equipped with the norm $\|\{a_n\}\|_\infty = \sup_n |a_n|$, and that $l_1(\mathbb{C})$ is the Banach space of sequences $\{a_n\}$ of complex numbers such that the sum $\sum_n |a_n|$ converges, equipped with the norm $\|\{a_n\}\|_1 = \sum_n |a_n|$.

- **(a)** (10 points). Prove that $l_1(\mathbb{C})$ is separable, but that $l_\infty(\mathbb{C})$ is not.
- **(b)** (5 points). Prove that $l_1(\mathbb{C})^* = l_\infty(\mathbb{C})$.
- **(c)** (10 points). Prove that $l_\infty(\mathbb{C})^* \neq l_1(\mathbb{C})$.

**7** (5 points). Evaluate

$$\int \cot^2(\pi z)\,dz$$

over the circle with center 0 and radius 1/2, described once counterclockwise.

**8** (10 points). Let $H = \{x \in \mathbb{C} : \Im(z) > 0\}$ and let $\bar{H} = H \cup \mathbb{R}$ be its closure. Let $f$ be a bounded and continuous function in $\bar{H}$, which is holomorphic in $H$ and such that $f(z)$ is real for all $z \in \mathbb{R}$. Prove that $f$ is constant.