Analysis Prelim Exam Spring 2009

Print Your Code:

Please state any theorem you use clearly. Good Luck!!

1. Prove using the definition of compactness only that a compact metric space has a countable dense subset. Recall that a set is compact if any open cover of the set has a finite subcover.

2. Let $\{f_n\}$ be a sequence of real valued functions on an open set $U \subset \mathbb{R}^2$. Suppose that each f_n has continuous first partial derivatives and that there are constants A, B, C such that

$$|f_n(x,y)| \leq A, \quad \left|\frac{\partial f_n}{\partial x}\right| \leq B, \quad \left|\frac{\partial f_n}{\partial y}\right| \leq C$$

for all n and all $(x,y) \in U$. Let K be a compact subset in U. Show that $\{f_n\}$ has a subsequence that converges uniformly on K.

3. a. Let $f \in L^p(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$, for some p and q satisfying $1 \leq p \leq q \leq \infty$. Show that for all $r \in [p,q]$ it follows that

$$\|f\|_r \leq \|f\|_p^{\theta} \|f\|_q^{1-\theta}, \text{ where } \frac{1}{r} = \frac{1}{p}\theta + \frac{1}{q}(1-\theta).$$

b. Let $f \in L^2 \cap L^\infty(\mathbb{R}^n)$. Show that

$$\lim_{p \to \infty} \|f\|_p = \|f\|_\infty.$$

4. Suppose that μ is a complex measure on a σ-algebra $\mathcal{M}$ in X. Then there is a measurable function h such that $|h(x)| = 1$ almost everywhere in $x \in X$ and

$$d\mu = h d|\mu|.$$

It might be helpful to remember the following:
Suppose that $\lambda(X) < \infty$, $f \in L^1(\lambda)$ and S a closed set in the complex plane. Suppose also that, for any λ-measurable set E with $\lambda(E) > 0$, the average of f over E lies in S. Then $f(x) \in S$ for almost all $x \in X$.

5. a. Let z_0 be in the upper half plane. Show that $w = e^{i\theta}\left(\frac{z-z_0}{z-\bar{z}_0}\right)$ maps the upper half plane into the interior of the unit disk.

b. Find a conformal mapping that maps the upper half plane into the interior of the unit circle so that $z = i$ is mapped into $w = 0$, and the point at infinity is mapped into $w = -1$

6. Find the Principal Value of the integral

$$\int_{-\infty}^{\infty} \frac{x \sin x \, dx}{x^2 + 2x + 2}$$

7. Suppose X is a Banach space. Show that the closed unit ball $B = \{x \in X : \|x\| \leq 1\}$ is closed in the weak topology.

8. Suppose X and Y are Banach spaces and that T is a compact linear operator from X to Y. Suppose also that

$$x_n \rightharpoonup x, \ \in X$$

weakly. Show that

$$Tx_n \to Tx, \ \in X$$

strongly.