Preliminary Exam Analysis – Spring 2010

6/7/10

Please state any theorem you use. Good Luck !!

1. Let $\{x_n\}$ and $\{y_n\}$ be Cauchy sequences in a metric space (X, d). Show that the sequence $\{d(x_n, y_n)\}$ converges.

2. Let X and Y be topological spaces and $X = \bigcup_{i=1}^n A_i$ where each A_i is closed. Show that if $f : X \to Y$ and each $f|_{A_i} : A_i \to Y$ is continuous, then so is f.

3. Let $\{f_n\}$ be a sequence of Lebesgue measurable functions defined on a set $E \subset \mathbb{R}$ of finite Lebesgue measure. Show that f_n converges to zero in measure if and only if
$$\int_E \frac{|f_n|}{1+|f_n|}\, dm \to 0.$$

4. Show: A function $f : \mathbb{R} \to \mathbb{R}$ is of bounded variation if and only if there exists a monotonically increasing function g such that for all $x_1 < x_2$ the following holds:
$$f(x_2) - f(x_1) \leq g(x_2) - g(x_1).$$
[For the "only if" part, it suffices to explain how to find the function g. The properties of g, which are used, should be stated, but no detailed proof of these properties is necessary.]

5. Let H be a Hilbert space and A be a bounded linear operator acting on H. Show: There exists a bounded linear operator B acting on H with the property that $BA = I$ if and only if there exists a constant $\gamma > 0$ such that
$$\|x\| \leq \gamma \|Ax\| \quad \forall x \in H.$$

6. Let X be a Banach space and X_1, X_2 be two closed subspaces such that $X = X_1 \dot{+} X_2$, i.e., each $x \in X$ can be written uniquely as a sum $x = x_1 + x_2$ with $x_1 \in X_1$ and $x_2 \in X_2$. Then the projection mapping $P : x \mapsto x_1$ is continuous. [Hint: use the Closed Graph Theorem]

7. Evaluate, using complex integration and residues,
$$\int_0^\infty \frac{1}{1+x^3}\, dx.$$

8. Determine all rational functions which map $\mathbb{C} \setminus \{0\}$ one-to-one into $\mathbb{C} \setminus \{0\}$.