Please state any theorem you use. Good Luck!!

1. Let $J$ be a countably infinite index set. Show that $X = [0, 1]^J$ with the product topology is metrizable.

2. Let $X$ be a compact Hausdorff space. Show that if $K$ and $L$ are closed disjoint subsets of $X$, then there exists open disjoint subsets $U \supseteq K$ and $V \supseteq L$.

3. Let $(X_1, \Lambda_1, \mu_1)$ and $(X_2, \Lambda_2, \mu_2)$ be two $\sigma$-finite measure spaces. Let $X = X_1 \times X_2$ and $\Lambda$ be the $\sigma$-algebra generated by all the rectangles $A \times B$ with $A \in \Lambda_1$, $B \in \Lambda_2$.
   Suppose that $\Phi$ and $\Psi$ are two measures defined on $\Lambda$ such that
   \[
   \Phi(A \times B) = \Psi(A \times B) = \mu_1(A)\mu_2(B), \quad A \in \Lambda_1, B \in \Lambda_2.
   \]
   Show that $\Phi = \Psi$.

4. Show the generalized Hölder inequality
   \[
   \left(\int |fg|^r \, dm\right)^{\frac{1}{r}} \leq \left(\int |f|^p \, dm\right)^{\frac{1}{p}} \cdot \left(\int |g|^q \, dm\right)^{\frac{1}{q}}
   \]
   for
   \[
   \frac{1}{p} + \frac{1}{q} = \frac{1}{r} \quad \text{and} \quad p, q, r > 0.
   \]

5. Show that $\int_0^\infty \frac{\sin(x)}{x} \, dx = \frac{\pi}{2}$.

6. Let $f$ be continuous on a domain $\Omega \subseteq \mathbb{C}$ and assume that $\int_T f(z) \, dz = 0$ for each triangle $T \subset \Omega$. Show that $f$ is holomorphic.

7. Let $T : \ell^2 \to \ell^2$ be defined by $T(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots)$ (right shift operator).
   (a) Show that $T$ has no eigenvalues.
   (b) Show that $\sigma(T) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$.
   (c) Determine all $\lambda \in \sigma(T)$ for which the range of $T - \lambda I$ is not dense in $\ell^2$.

8. Let $C$ be a convex set in a real normed linear space $X$ such that $0 \in C$. Show that for each $x_0 \notin C$ there exists a continuous linear functional $F$ such that $F(x) \leq F(x_0)$ for all $x \in C$. 