Preliminary Exam Analysis – Spring 2012

6/13/2012

Please state any theorem you use. Good Luck !!

1. Let J be a countably infinite index set. Show that $X = [0,1]^J$ with the product topology is metrizable.

2. Let X be a compact Haussdorff space. Show that if K and L are closed disjoint subsets of X, then there exists open disjoints subsets $U \supseteq K$ and $V \supseteq L$.

3. Let (X_1, Λ_1, μ_1) and (X_2, Λ_2, μ_2) be two σ-finite measure spaces. Let $X = X_1 \times X_2$ and Λ be the σ-algebra generated by all the rectangles $A \times B$ with $A \in \Lambda_1$, $B \in \Lambda_2$.
   Suppose that Φ and Ψ are two measures defined on Λ such that
   $$\Phi(A \times B) = \Psi(A \times B) = \mu_1(A)\mu_2(B), \qquad A \in \Lambda_1, B \in \Lambda_2.$$
   Show that $\Phi = \Psi$.

4. Show the generalized Hölder inequality
   $$\left(\int_{\mathbb{R}} |fg|^r dm\right)^{\frac{1}{r}} \leq \left(\int_{\mathbb{R}} |f|^p dm\right)^{\frac{1}{p}} \cdot \left(\int_{\mathbb{R}} |g|^q dm\right)^{\frac{1}{q}}$$
   for
   $$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} \qquad \text{and } p, q, r > 0.$$

5. Show that $\int_0^\infty \frac{\sin(x)}{x}\, dx = \frac{\pi}{2}$.

6. Let f be continuous on a domain $\Omega \subseteq \mathbb{C}$ and assume that $\int_T f(z)\, dz = 0$ for each triangle $T \subset \Omega$. Show that f is holomorphic.

7. Let $T : \ell^2 \to \ell^2$ be defined by $T(x_1, x_2, \dots) = (0, x_1, x_2, \dots)$ (right shift operator).
   (a) Show that T has no eigenvalues.
   (b) Show that $\sigma(T) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$.
   (c) Determine all $\lambda \in \sigma(T)$ for which the range of $T - \lambda I$ is not dense in ℓ^2.

8. Let C be a convex set in a real normed linear space X such that $0 \in C$. Show that for each $x_0 \notin C$ there exists a continuous linear functional F such that $F(x) \leq F(x_0)$ for all $x \in C$.