Preliminary Exam Analysis – Spring 2014

5/23/2014

Please state any theorem you use. Good Luck !!

1. Let (X, d) be a metric space, $A \subset X$ be a closed subset, and $p \notin A$. Show that there exist open subsets U and V in X such that $p \in U$, $A \subset V$, and $U \cap V = \emptyset$.

2. Let $C(X)$ be the space of continuous functions from a compact metric space X into $\mathbb{R}$ with the sup-norm
$$\|f\|_{C(X)} = \max_{x \in X} |f(x)|.$$
Suppose that $K \subset C(X)$ is compact. Show that functions in K are uniformly bounded and uniformly equi-continuous.

3. Let f be a real-valued function of bounded variation on $[0, 1]$. Show that the only discontinuities of f are jump discontinuities and that f has at most countably many of them.

4. Let f_n be real-valued, Lebesgue measurable functions defined $[0, 1]$ which converge a.e. to a real-valued function f. Show that f is Lebesgue measurable.

5. Let X be a normed space. Show that a linear operator $A : X \to X$ is continuous if and only if it is bounded.

6. Let X be a Banach space. Show that the set $G \subseteq L(X, X)$ of all invertible bounded linear operators is an open set.

7. Evaluate, for $a > 0$,
$$\int_0^\infty \frac{\log x}{x^2 + a^2}\, dx.$$

8. Let $p_n(z) = z/1! + z^2/2! + z^3/3! + \cdots + z^n/n!$, and let $Z(n; R)$ stand for the number of zeros of $p_n(z)$, taking multiplicities into account, in the disk $\{\, z \in \mathbb{C} \,:\, |z| < R \,\}$. Determine the limit
$$\lim_{n \to \infty} Z(n; 5\pi).$$