Analysis Preliminary Exam, Math @ UCSC, Spring 2015

1. Suppose that $f_n(x)$ uniformly converges to $f(x)$ on the interval $[a,b]$. Assume also that, for $x_0 \in (a,b)$,
$$\lim_{x\to x_0} f_n(x) = a_n$$
for each n. Show that $\lim_{x\to x_0} f(x)$ exists and
$$\lim_{x\to x_0} f(x) = \lim_{n\to\infty} a_n$$
i.e.
$$\lim_{x\to x_0}\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}\lim_{x\to x_0} f_n(x).$$

2. Suppose $f_n(x)$ is a sequence of continuously differentiable functions on a finite interval $[a,b]$. Assume $f_n(x)$ and the derivatives $f_n'(x)$ are uniformly bounded on the interval $[a,b]$. Show that there always exists a subsequence of $f_n(x)$ that uniformly converges on the interval $[a,b]$.

3. Construct a function such that each set $\{f(x)=\alpha\}$ is Lebesgue measurable for any $\alpha\in\mathbb{R}$, but the set $\{f(x)>0\}$ is not Lebesgue measurable.

4. Show that if $\mu(E)<\infty$ and a sequence of measurable functions $f_n \to f \in L^1_\mu(E)$ a.e. on E as $n\to\infty$, then the following are equivalent:

   (1) f_n are uniformly integrable, namely, for any $\epsilon>0$ there exists $\delta=\delta(\epsilon)$ such that $\int_{E\cap\{|f_n|\geq\delta\}} |f_n| \leq \epsilon$;

   (2) $\int_E |f_n - f| \to 0$;

   (3) $\int_E |f_n| \to \int_E |f|$.

5. Let X be a Banach space with norm $\|\cdot\|$. Assume that there is a second norm $\|\cdot\|_2$ defined on X under which X is also complete and that we have $\|x\| \leq \|x\|_2$ for all $x\in X$. Show that there exists $c>0$ such that $\|x\|_2 \leq c\|x\|$ for all $x\in X$.

6. Let X be a reflexive Banach space and $Y\subset X$ be a closed subspace. Show that Y as a Banach space is also reflexive.

7. Let f be holomorphic and nonzero on $\Omega=\mathbb{C}\setminus\{0\}$ and assume that
$$\int_{|z|=1} \frac{f'(z)}{f(z)}\,dz = 0.$$
Show that $f(z)$ possesses a holomorphic logarithm on Ω.

8. Let f_n be a sequence of entire holomorphic functions converging uniformly on every compact subset of $\mathbb{C} \setminus \mathbb{R}$. Assume in addition that

$$|f_n(z)| \leq \frac{1}{|\mathrm{Im}(z)|^{1/2}}$$

for all $z \in \mathbb{C} \setminus \mathbb{R}$ and all n. Show that the limit function f is entire and that the convergence $f_n \to f$ is uniform on every compact subset of $\mathbb{C}$.