Spring 2018 - Analysis prelim - Friday, June 8, 9am-2pm
University of California Santa Cruz

1. Suppose $f_n \to f$ uniformly and $\lim_{x\to x_0} f_n(x)$ exists for every n. Then $\lim_{x\to x_0} f(x)$ exists and equals $\lim_{n\to\infty} \lim_{x\to x_0} f_n(x)$.

2. Let $\{f_n\}$ be a sequence of twice differentiable functions on $[0,1]$ such that $f_n(0) = f_n'(0) = 0$ for all n and such that $|f_n''(x)| \le 1$ for all n and all $x \in [0,1]$. Show that there exists a subsequence of $\{f_n\}$ which converges uniformly on $[0,1]$.

3. (a) Show that if $f \in L^1(\mathbb{R})$ and f is uniformly continuous, then $\lim_{x\to\infty} f(x) = 0$.
   (b) Does the conclusion still hold without assuming uniform continuity ? Justify your answer.

4. Let m denote Lebesgue measure on $\mathbb{R}$. Let $f_n : \mathbb{R} \to \overline{\mathbb{R}}$ be a sequence of measurable functions such that $f_n \to f$ in measure[1]. Suppose there is an integrable function g such that $|f_n(x)| \le g(x)$ for a.e. $x \in \mathbb{R}$.
   (a) Show that $|f(x)| \le g(x)$ for a.e. $x \in \mathbb{R}$.
   (b) Show that $\lim_{n\to\infty} \int_{\mathbb{R}} |f_n - f| = 0$.

5. Let X be a Banach space and K be a compact linear operator on X. Assume (for simplicity) that $A = I + K$ has a trivial nullspace. Show that the range of A is closed.

6. Let X and Y two Banach spaces. A sequence of bounded linear operators $A_n \in L(X,Y)$ is said to converge weakly to a linear operator A (from X to Y) if for all $x \in X$ and all $\phi \in Y^*$ the sequence $\phi(A_n x)$ converges to $\phi(Ax)$. Assuming A_n converges weakly to A, show that $\sup_n \|A_n\| < \infty$ and that the operator A is bounded.

7. (a) State Rouché's theorem.
   (b) Prove that any polynomial with complex coefficients $P(z) = \sum_{k=0}^n a_k z^k$ of degree n ($a_n \neq 0$) has exactly n roots, and give an estimate on the radius of the disk $D_R(0)$ containing all roots.

8. Show using residue theory that for every $\xi \in \mathbb{R}$,

$$\int_{-\infty}^{\infty} \frac{e^{-2\pi i x\xi}}{\cosh(\pi x)}\, dx = \frac{1}{\cosh(\pi\xi)}.$$

   Justify rigorously any limiting process involved.

[1] f_n converges to f in measure if for every $\varepsilon > 0$, $\lim_{k\to\infty} m(\{x : |f_n(x) - f(x)| > \varepsilon\}) = 0$