Spring 2019 - Analysis prelim - Friday, May 31
University of California Santa Cruz

1. Show that the mapping $T : \mathbb{R} \to \mathbb{R}$ defined by

$$T(x) = \frac{\pi}{2} + x - \arctan(x)$$

satisfies $|T(x) - T(y)| \leq |x - y|$ for all $x, y \in \mathbb{R}$, but has no fixed point in $\mathbb{R}$. State the Contraction Mapping Theorem and explain why does this example not contradict this theorem.

2. Suppose that $U \subset \mathbb{R}^n$ is an open set, $K \subset U$ and K is compact. Prove that there is an open set V whose closure is compact such that

$$K \subset V \subset \overline{V} \subset U\,.$$

Please state clearly the statements of the theorems you are using and verify the assumptions needed to apply the theorems.

3. Prove that a Lipshitz function $f : \mathbb{R} \to \mathbb{R}$ maps sets of Lebesgue measure zero to sets of Lebesgue measure zero. For which values of n and m does the same statement hold for Lipshitz functions $f : \mathbb{R}^m \to \mathbb{R}^n$? (Provide proofs or counterexamples.)

4. Let μ and ν be two σ-finite measures on a σ-algebra $\mathcal{F}$ of subsets of a set X such that $\nu \ll \mu$ (ν is absolutely continuous with respect to μ), and $\nu \neq 0$. Show that there exist a set $E \in \mathcal{F}$ and an integer $n > 0$ such that $\nu(E) > 0$ and $(1/n)\mu(A) \leq \nu(A) \leq n\mu(A)$ for all $A \in \mathcal{F}$ with $A \subseteq E$.

5. Let X be a Banach space and $A \in L(X)$ be a bounded linear operator. Show that there exists a bounded linear operator $B \in L(X)$ such that $AB = BA = I_X$ if and only if there exists a constant $\gamma > 0$ such that

$$\|x\| \leq \gamma \|Ax\|, \qquad \|\phi\| \leq \gamma \|A^*\phi\| \qquad \text{for all } x \in X,\ \phi \in X^*.$$

Here A^* stands for the Banach space adjoint of A and I_X is the identity operator on X.

6. Show that every closed linear subspace of a reflexive Banach space is reflexive.

7. Using complex analysis, evaluate the integrals

$$I_1 = \int_0^\infty \frac{1 - \cos x}{x^2}\, dx, \qquad I_2 = \int_0^{2\pi} \frac{1}{2 + \cos\theta}\, d\theta.$$

8. Let $s > 0$ be a real number. Describe the set of all functions f holomorphic on $\Omega = \mathbb{C} \setminus \{0\}$ for which there exists a constant $A > 0$ such that $|f(z)| \leq A(1 + |z|^{-s})$ for all $z \in \Omega$. Prove your statement.