PRELIMINARY EXAMINATION 2005 DECEMBER 2

1. Let X be a metric space, and let A and B be disjoint closed subsets in X. Show that there is a continuous function $f : X \to [0,1]$ such that $f = 0$ on A and $f = 1$ on B.

2. Let X be a complete metric space and $\{f_\alpha\}$ be a collection of real-valued continuous functions on X. Suppose that, for each $x \in X$,

$$\sup_\alpha f_\alpha(x) < \infty.$$

Show that there is an open subset $U \subset X$ and a number $M > 0$ such that

$$\sup_\alpha f_\alpha(x) < M, \forall x \in U.$$

3. Let V be a normed vector space and W a closed subspace of V. Suppose that $x_0 \in V$ and $\phi(x_0) = 0$ for all continuous linear functionals ϕ whose kernels contain W. Show that $x_0 \in W$.

4. Let V be a Banach space, and for $a > 0$ let

$$B_a = \{x^* \in V^* : \|x^*\| \leq a\}.$$

Prove that B_a is also closed in the weak* topology.

5. Evaluate the intregral

$$\int_{-\infty}^{\infty} \frac{1}{x^4+1} dx.$$

6. Let f be an entire analytic function and n a positive integer. Suppose that

$$\frac{f(z)}{1+|z|^n}$$

is bounded. Show that f is a polynomial of degree at most n.

7. Suppose that $f_n \to f$ in $L^1(mu)$ and $f_n \to g$ a. e. (μ). What can you say about the relation between f and g? Justify your answer!

8. Show that if $f \in L^2(0,1)$ then the function

$$g(x) = \frac{1}{x}\int_0^x f(y)dy$$

belong to $L^1(0,1)$.