1. A topological space is said to be sequentially compact if any sequence of points from the space has a convergent subsequence. Show that the product of two sequentially compact topological spaces is sequentially compact.

2. Let $X$ be a complete metric space and \( \{A_n\} \) be a sequence of nonempty closed subsets such that

\[
A_{n+1} \subset A_n \quad \text{and} \quad \lim_{n \to \infty} \text{diam}(A_n) = 0.
\]

Recall that

\[
\text{diam}(A) = \sup\{d(x, y) : x, y \in A\}.
\]

Prove that \( \bigcap_{n=1}^{\infty} A_n \) is nonempty.

3. 
   a. State the Carathéodory criterion for a subset \( A \subset \mathbb{R} \) of the real numbers to be Lebesgue measurable.
   b. Prove that an open interval \((a, +\infty) \subset \mathbb{R}\) is Lebesgue measurable.

4. Let \( \{f_1(x), f_2(x), \ldots\} \) be a sequence of nonnegative functions from a Lebesgue measurable subset \( E \) of the reals.
   a. State the Fatou Lemma.
   b. Suppose that \( f_i(x) \to f(x) \) almost everywhere in \( E \) and that

\[
\lim_{i \to \infty} \int_E f_i(x) \, dm(x) = \int_E f(x) \, dm(x) < \infty.
\]

Show that, for any Lebesgue measurable subset \( A \subset E \),

\[
\lim_{i \to \infty} \int_A f_i(x) \, dm(x) = \int_A f(x) \, dm(x).
\]

5. Suppose that \( \{f_n\} \) are analytic functions on an open domain \( D \subset \mathbb{C} \) and that \( f_n \) uniformly converge to a function \( f \) in \( D \). Show that \( f \) is also analytic in \( D \).

6. Compute the integral

\[
\int_0^\infty \frac{\cos x}{1 + x^2} \, dx.
\]
7. Suppose that $X, Y$ are Banach spaces and that $A \in L(X, Y)$. Let $X', Y'$ are the dual space of $X, Y$ respectively.
   a. Define the conjugate operator $A'$ of $A$ from $Y'$ to $X'$.
   b. Show that $A'$ is a bounded linear operator from $Y'$ to $X'$ and $\|A'\| = \|A\|$.

8. Let $X$ be a Banach space and $A \in L(X)$ be a bounded linear operator.
   a. Define the resolvent set of $A$ and the spectral radius of $A$.
   b. Show that the spectral radius of $A$ is less than or equal to $\|A\|$.