ANALYSIS PRELIMINARY EXAMINATION DEC 2006

1. A topological space is said to be sequentially compact if any sequence of points from the space has a convergent subsequence. Show that the product of two sequentially compact topological spaces is sequentially compact.

2. Let X be a complete metric space and $\{A_n\}$ be a sequence of nonempty closed subsets such that

$$A_{n+1} \subset A_n \quad \text{and} \quad \lim_{n\to\infty} \operatorname{diam}(A_n) = 0.$$

Recall that

$$\operatorname{diam}(A) = \sup\{d(x,y) : x, y \in A\}.$$

Prove that $\bigcap_{n=1}^{\infty} A_n$ is nonempty.

3.

a. State the Carathéodory criterion for a subset $A \subset R$ of the real numbers to be Lebesgue measurable.

b. Prove that an open interval $(a, +\infty) \subset R$ is Lebesgue measurable.

4. Let $\{f_1(x), f_2(x), \cdots\}$ be a sequence of nonnegative functions from a Lebesgue measurable subset E of the reals.

a. State the Fatou Lemma.

b. Suppose that $f_i(x) \to f(x)$ almost everywhere in E and that

$$\lim_{i\to\infty} \int_E f_i(x)dm(x) = \int_E f(x)dm(x) < \infty.$$

Show that, for any Lebesgue measurable subset $A \subset E$,

$$\lim_{i\to\infty} \int_A f_i(x)dm(x) = \int_A f(x)dm(x).$$

5. Suppose that $\{f_n\}$ are analytic functions on an open domain $D \subset C$ and that f_n uniformly converge to a function f in D. Show that f is also analytic in D.

6. Compute the integral

$$\int_0^{\infty} \frac{\cos x}{1+x^2} dx.$$

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7. Suppose that X, Y are Banach spaces and that $A \in L(X, Y)$. Let X', Y' are the dual space of X, Y respectively.

a. Define the conjugate operator A' of A from Y' to X'.

b. Show that A' is a bounded linear operator from Y' to X' and $||A'|| = ||A||$.

8. Let X be a Banach space and $A \in L(X)$ be a bounded linear operator.

a. Define the resolvent set of A and the spectral radius of A.

b. Show that the spectral radius of A is less than or equal to $||A||$.