Preliminary Exam Analysis – Winter 2010

1/7/10

Please state any theorem you use. Good Luck !!

1. Suppose that X and Y are nonempty topological spaces such that $X \times Y$ is Hausdorff. Show that X and Y are Hausdorff.

2. (a) Let X be a compact topological space, Y a Hausdorff space, and f a continuous bijective function from X to Y. Show that f is a homeomorphism.

   (b) Give an example where Y is not Hausdorff, while the other assumptions in part (a) are satisfied, and the conclusion fails.

3. Let f_n be a sequence of integrable functions converging a.e. (on $\mathbb{R}$) to an integrable function f. Show that $\int_{\mathbb{R}} |f_n - f|\, d\mu \to 0$ if and only if $\int_{\mathbb{R}} |f_n|\, d\mu \to \int_{\mathbb{R}} |f|\, d\mu$. (Here μ is the Lebesgue measure.)

4. Let $(X, \mathcal{B})$ be a measurable space, and let μ_n be a sequence of measures on $\mathcal{B}$ such that $\mu_{n+1}(E) \geq \mu_n(E)$ for each $E \in \mathcal{B}$. Show that μ defined by $\mu(E) = \lim \mu_n(E)$ is a measure.

5. Assume that X is a Banach space and $M \neq X$ is a closed linear subspace of X. Show that there exists an element $x_0 \in X$ of unit length such that
$$\operatorname{dist}(x_0, M) \geq \frac{1}{2}.$$

6. Let ℓ^2 be the Hilbert space of square-summable sequences $\{x_i\}_{i=1}^{\infty}$ of real numbers equipped with the norm
$$\|\{x_i\}\| := \Big(\sum_{i=1}^{\infty} |x_i|^2\Big)^{1/2} < \infty.$$
   The shift operator T on ℓ^2 is defined by
$$T(\{x_1, x_2, x_3, \dots\}) = \{x_2, x_3, x_4, \dots\}.$$
   Show that T is a Fredholm operator of index 1.

7. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire bijective function. Prove that f is linear, that is $f(z) = az + b$ for some constants a and b.

8. (a) Show that a sufficient condition for the infinite product $\Pi_{k=1}^{\infty}(1 + a_k)$ to converge is that the series $\sum_{k=1}^{\infty} |a_k|$ converges.

   (b) Prove that $\Pi_{k=1}^{\infty}\left(1 + \frac{z}{k}\right) e^{-\frac{z}{k}}$ converges for each $z \in \mathbb{C}$.