Preliminary Exam Analysis – Winter 2011

1/14/11

Please state any theorem you use. Good Luck !!

1. Let X be a metric space and Y a subspace of X such that Y is complete. Show that Y is closed.

2. Let J be an uncountable index set, and let $X = [0,1]^J$ with product topology. Show that X is not metrizable.

3. Consider the measurable space $(\mathbb{R}_+, \mathfrak{B})$, with $\mathbb{R}_+ = (0,\infty)$ and $\mathfrak{B}$ the class of Borel sets. Let μ be a measure (on this measurable space). Assume that for each $\xi \geq 0$ the integral
$$f(\xi) := \int_{\mathbb{R}_+} x^\xi \, d\mu(x)$$
is finite. Show that $f(\xi)$ is differentiable for $\xi \in (0,\infty)$ and continuous at $\xi = 0$.

4. Let f and g be absolutely continuous functions on $[0,1]$. Then their product is also absolutely continuous.

5. Let V be a normed space. Give the definition of the natural embedding τ of V into its second dual V^{**}. Show that τ is an isometry.

6. Let V be a complex vector space with the inner product $< \cdot, \cdot >$. Prove the Cauchy-Schwartz inequality.

7. Prove that
$$\int_0^\infty \frac{\sin(x)}{x} \, dx = \frac{\pi}{2}.$$

8. How many roots of the equation $z^7 - 2z^5 + 6z^3 - z + 1 = 0$ lie in the disk $|z| < 1$?