Preliminary Exam Analysis – Winter 2012

1/20/2012

Please state any theorem you use. Good Luck !!

1. (a) Let $D \subseteq \mathbb{R}^n$ be an open set and $S \subseteq D$ be a subset which has no accumulation points in D. Prove that S is at most countable.
(b) Does the statement still hold if D is replaced by an arbitrary topological space which has a countable basis ?

2. Show that a compact metric space is a complete metric space.

3. Let $(S, \mathfrak{B})$ be a measurable space, and μ_n a sequence of (positive) measures on this space. Show that $\mu(A) := \sum_{n=1}^{\infty} \mu(A)$, $A \in \mathfrak{B}$, defines a measure.

4. Let $f : [0,1] \to \mathbb{R}$ be an integrable function. Show that $F(x) = \int_0^x f(t)\,dt$ is an absolutely continuous function.

5. Let $\Omega = \{z \in \mathbb{C} : |z| < 1\}$ and let f_n be a sequence of holomorphic functions such f_n converges uniformly on Ω. Does it follow that the derivatives f_n' converge uniformly on Ω ? What about the convergence of f_n' on compact subsets of Ω ?

6. Show that the infinite product

$$g(z) = e^{z\gamma} \prod_{k=1}^{\infty} \left(1 + \frac{z}{k}\right) e^{-z/k}$$

converges to an entire function, where γ is Euler's constant. Prove the recurrence

$$(z+1)g(z+1) = g(z).$$

7. Let $H = L^2[0,1]$ and $T : H \to H$ be the Volterra integral operator

$$(Tf)(x) = \int_0^x f(t)\,dt, \qquad 0 < x < 1,$$

(a) Show that T has no nonzero eigenvalues.
(b) Compute the adjoint of of T.

8. Suppose E is a separable Banach space. Show that the unit ball B^* in E^* is compact in the weak-* topology.