ANALYSIS PRELIM. JAN 25 2013

1. Let $f : X \to Y$ be a continuous, bijective map, where X is a compact topological space and Y is a Hausdorff space. Show that f is a homeomorphism.

2. Let $X = C(\mathbb{R})$ stand for the set of all continuous functions $f : \mathbb{R} \to \mathbb{R}$. Equip X with the metric

$$d(f,g) = \sum_{N=1}^{\infty} \frac{1}{2^N} \frac{d_N(f,g)}{1+d_N(f,g)} \text{ with } d_N(f,g) = \max_{|x|\le N} |f(x) - g(x)|.$$

Show that the metric space (X, d) is complete.

3. Use complex analysis to compute

$$\int_0^\infty \frac{\sin x}{x} dx.$$

4. Suppose $f_n : \mathbb{C} \to \mathbb{C}$ is a sequence of entire functions and f_n uniformly converges to a function $f : \mathbb{C} \to \mathbb{C}$ on any compact subset of the complex plan $\mathbb{C}$. Show that f is an entire function too.

5. Suppose that $A_1 \supset A_2 \supset A_3 \cdots$ is a sequence of subsets of a measure space (X, Λ, μ).

a) Assume that $\mu(A_1) < \infty$. Then show that

$$\lim_{i\to\infty} \mu(A_i) = \mu(\bigcap_{i+1}^{\infty} A_i).$$

b) If the assumption $\mu(A_1) < \infty$ is dropped, can you prove that

$$\lim_{i\to\infty} \mu(A_i) = \mu(\bigcap_{i+1}^{\infty} A_i)?$$

6. Let (X, Λ, μ) be a measure space. Suppose that A_i is a measurable set with

$$\mu(A_i) \leq 2^{-i}$$

for each $i = 1, 2, 3, \cdots$. Show that

$$\mu(\bigcap_{k=1}^{\infty} \bigcup_{i=k}^{\infty} A_i) = 0.$$

7. Let

$$l^p = \{(x_1, x_2, x_3, \cdots) : x_i \in \mathbb{R} \text{ and } \sum_{i=1}^{\infty} |x_i|^p < \infty\} \text{ for } 1 \leq p < \infty \text{ and}$$
$$l^\infty = \{(x_1, x_2, x_3, \cdots) : x_i \in \mathbb{R} \text{ and } \sup_i |x_i| < \infty\}$$

where

$$\|(x_1, x_2, x_3, \cdots)\|_{l^p} = (\sum_{i=1}^{\infty} |x_i|^p)^{\frac{1}{p}} \text{ for } 1 \leq p < \infty \text{ and}$$
$$\|(x_1, x_2, x_3, \cdots)\|_{l^\infty} = \sup_i |x_i|.$$

a) Let

$$\mathrm{T}(\xi_1, \xi_2, \xi_3 \cdots) = (\xi_1, \frac{1}{2}\xi_2, \frac{1}{3}\xi_3, \cdots) : l^2 \to l^2.$$

Prove T is compact

b) Let

$$U(x_1, x_2, x_3, \cdots) = (0, x_1, 0, x_3, \cdots) : l^p \to l^p$$

for any $1 \leq p \leq \infty$. Show that U is not a compact operator, although $U^2 = 0$.

8. Let X be a normed vector space and let $x_0 \neq 0$ be an element of X. Show that there is a bounded linear functional $F : \mathrm{X} \to \mathbb{R}$ satisfying

$$\|F\| = 1, \;\; F(x_0) = \|x_0\|.$$