Preliminary Exam Analysis – Winter 2014

1/17/2014

Please state any theorem you use. Good Luck !!

1. Suppose that $\{f_n\}_{n=1}^{\infty}$ is a sequence of monotonically increasing real-valued continuous functions on $[0,1]$. Assume that f_n converges pointwise to a continuous function f on $[0,1]$. Show that the family $\{f_n\}_{n=1}^{\infty}$ is uniformly equi-continuous on $[0,1]$.

2. Suppose that a sequence of real numbers x_n converges to a real number x_0. Show that

$$\lim_{n\to\infty} \frac{x_1 + x_2 + \cdots + x_n}{n} = x.$$

3. Let $f : [0,1]^2 \to \mathbb{R}$ be a function such that $f(x,y)$ is Lebesgue integrable in x for each fixed y, and differentiable in y for each fixed x. Assume there is a Lebesgue integrable function $g(x)$ such that $\left|\frac{\partial f}{\partial y}(x,y)\right| \leq g(x)$ for every x, y. Show that $\psi(y) := \int_0^1 f(x,y)\,dx$ can be differentiated under the integral sign.

4. Let μ and ν be finite nonnegative measures on a measure space $(X, \mathcal{Q})$, such that $\mu \ll \nu$. Let $\frac{d\nu}{d(\mu+\nu)}$ stand for the Radon-Nykodim derivative of ν with respect to $\mu + \nu$. Show that

$$0 < \frac{d\nu}{d(\mu+\nu)} < 1 \quad [\mu]\text{-a.e.}$$

5. Suppose that X is a normed vector space and that W is a closed subspace in X. Let $x_0 \in X$. Assume that $\phi(x_0) = 0$ for all $\phi \in X^*$ with $N(\phi) \supseteq W$. Show that $x_0 \in W$.

6. Suppose that $C(\Omega)$ is the space of all continuous functions from a bounded connected and open domain Ω in the Euclidean space $\mathbb{R}^n$. Define a vector topology on the space $C(\Omega)$ such that it becomes a Fréchet space. (Prove that it becomes a Fréchet space.)

7. State the Casorati-Weierstrass Theorem. Using it show that the only bi-holomorphic maps of $\mathbb{C}$ to itself are mappings of the form $f(z) = Az + B$.

8. Let Ω be a connected open domain in $\mathbb{C}$ and f be a holomorphic function on Ω which does not vanish identically. Show that the zeros of f are isolated.