Preliminary Examination Analysis 2015 Jan 23

(1). Assume that X is Heine-Borel compact, that is, every open cover has a finite subcover. Show that X is also sequentially compact, that is, every sequence of points in X has a limit point in X.

(2). Prove that the set Q of rational numbers can not be an intersection of a sequence of open sets.

(3). Let Ω be a connected open domain in the complex plane C and f be a non-constant analytic function on it. Show that the function $Re(f(z))$ cannot attain a minimum or maximum in Ω.

(4). Evaluate

$$\int_0^\infty \frac{1-\cos x}{x^2}\,dx$$

(5). Let E be a subset of $[0,1]$ of Lebesgue measure 1. Show that E is dense in $[0,1]$.

(6). Let g be an integrable function on a set E. And suppose $\{f_n\}$ is a sequence of measurable functions such that $|f_n| \le g, \forall n = 1, 2, \ldots$ a.e. on E. Show that

$$\int_{\mathrm{E}} \underline{\lim}_n f_n d\mu \le \underline{\lim}_n \int_{\mathrm{E}} f_n d\mu \le \overline{\lim}_n \int_{\mathrm{E}} f_n d\mu \le \int_{\mathrm{E}} \overline{\lim}_n f_n d\mu.$$

(7) Let $A : X \to Y$ be a bounded linear operator between normed spaces X and Y. Show that the nullspace (kernel) of A is a closed subspace of X.

(8) Let X be a vector space and let $\|.\|_1$ and $\|.\|_2$ be two norms on X such that each of it makes X not only a normed space but a Banach space. Assume that, for some constant $M > 0$,

$$\|x\|_1 \le M \|x\|_2$$

for all $x \in X$. Show that there exists a constant $m > 0$ such that

$$\|x\|_2 \le m \|x\|_1$$