Analysis Preliminary Exam, Math @ UCSC, Winter 2016

1. Suppose that (X, d) is a metric space and that A and B are disjoint closed subsets of X. Prove that there are two disjoint open subsets U and V of X such that $A \subset U$ and $B \subset V$

2. Suppose that (X, d) is a complete metric space. We say that a subset A of X is of finite ϵ-net property if, for every $\epsilon > 0$, there are finitely many points $\{x_i\}_{i=1}^k$ from A such that $A \subset \bigcup_{i=1}^k B_\epsilon(x_i)$, where each $B_\epsilon(x_i)$ is the ball centered at x_i with radius ϵ. Prove that a closed subset A of X is Heine-Borel compact if and only if it is of finite ϵ-net property.

3. Suppose that for some $p \in (1, \infty)$, $f_n \in L^p([0, 1])$ and $\|f_n\|_p \leq 1$, uniformly in n. Assuming that $f_n(x) \to 0$ a.e.; prove that $f_n \to 0$ weakly in L^p.

4. Let X be an uncountable set and $\mathcal{M}$ be the collection of all sets $E \subset X$ such that either E or E^c is at most countable. Define $\mu(E) = 0$ in the first case and $\mu(E) = 1$ in the second. Prove that $\mathcal{M}$ is a σ-algebra and that μ is a measure on $\mathcal{M}$.

5. Let $A : X \to Y$ be an invertible bounded linear operator between Banach spaces X and Y. Show

that there exists an $\varepsilon > 0$ such that for all bounded linear operators $B : X \to Y$ with $\|B\| < \varepsilon$, the operator $A + B$ is invertible.

6. Let $A : X \to Y$ be a bounded linear operator (where X and Y are Banach spaces). Define the adjoint A^* and show that $\|A\| = \|A^*\|$.

7. Let
$$G(z) = z \prod_{k=1}^{\infty} \left(1 + \frac{z}{k}\right) e^{-z/k} .$$
(a) Show that this infinite product defines an entire function.
(b) Where are the zeros of this function? Why?
(c) Compute (in terms of $G(z)$) the limit
$$\lim_{N\to\infty} \frac{z(z+1)(z+2)\cdots(z+N)}{N!} \cdot N^{-z} .$$

8. Show that
$$\int_{-\infty}^{\infty} \frac{e^{-2\pi i x\xi}}{\cosh \pi x}\, dx = \frac{1}{\cosh \pi\xi} .$$