Preliminary Exam Analysis – Winter 2017

1/20/2017

Please state any theorem you use. Good Luck !!

1. (a) Show that the image of a continuous function $f : X \to Y$ (where X and Y are topological spaces) is connected if X is connected.
   (b) Show that the product $X \times Y$ of two connected topological spaces X and Y is connected.

2. (a) Show that every closed subset of a compact space is compact.
   (b) Show that every compact subset of a Hausdorff space is closed.
   (c) Give an example of a T1-space X which has a non-closed, compact subset $A \subseteq X$.

3. Let f be a non-negative measurable function defined on the real line. Show that there exists a sequence ϕ_n of non-negative simple functions, each of which vanishes outside a set of finite measure, such that $f(x) = \lim_{n\to\infty} \phi_n(x)$ for every $x \in \mathbb{R}$.

4. Let f_n be sequence of measurable functions (defined on $\mathbb{R}$) such that $f_n \to f$ in measure. Suppose there is an integrable function g such that $|f_n| \leq g$. Show that $\int |f_n - f| \to 0$.

5. Let A be a subset in a normed vector space V. Suppose that A is weakly bounded, that is,

   $$\sup_{x\in A} |f(x)| < \infty$$

   for each $f \in V^*$. Prove that A is bounded, that is, $\sup_{x\in A} \|x\| < \infty$.

6. Let V and W be Banach spaces and let $T : V \to W$ be a closed injective linear operator. Then there is a number $c > 0$ such that $\|x\| \leq c\|Tx\|$ for all $x \in V$ if and only if the range $R(T)$ is closed.

7. Let f be a holomorphic function defined on $\{z \in \mathbb{C} : 0 < |z| < 1\}$ and suppose that the inequality $|f(z)| \leq C|z|^{-1/2}$ is satisfied. Show that f has a removable singularity at $z = 0$.

8. Evaluate

   $$\int_0^\infty \frac{\sin(x)}{x}\, dx.$$