Analysis Preliminary Exam, Math @ UCSC, Winter 2018

1. A set $A \subset \mathbf{R}$ is *perfect* if every point $x \in A$ is a limit point. Prove that every nonempty, closed, perfect set $S \subset \mathbf{R}$ is uncountable.

2. Let (X, ρ) be a metric space. A continuous function $f \in C(X)$ is said to be *Hölder continuous of exponent* $\alpha > 0$ if
$$H_\alpha(f) := \sup_{x \neq y} \frac{|f(x) - f(y)|}{\rho^\alpha(x,y)} < \infty\,.$$
Show that if X is compact then the set
$$A := \{f \in C(X) : \|f\|_{C(X)} \leq 1 \quad \text{and} \quad H_\alpha(f) \leq 1\}$$
is compact in $C(X)$.

3. Let $\{f_n\}$ be a sequence of measurable functions on $\mathbf{R}$, and $f_n \to f$ almost everywhere. Prove that there exists a sequence of measurable sets $\{E_k\}$ such that the Lebesgue measure of $\mathbf{R} \setminus (\cup_{k=1}^\infty E_k)$ is zero, and $f_n \to f$ uniformly on each E_k.

4. Let $\{f_n\}$ be a sequence of measurable functions on a complete measure space (X, Λ, μ) such that f_n converges to a measurable function f *in measure* μ. Suppose there exists $g \in L^1_\mu(X)$ such that $|f_n(x)| \leq g(x)$ μ-almost-everywhere for all n. Show that
$$f \in L^1_\mu(X) \quad \text{and} \quad \lim_{n\to\infty} \int_X |f_n - f| d\mu = 0\,.$$

5. Show that if the sequence $\{x_n\}$ in a normed space X is weakly convergent to $x_0 \in X$, then
$$\liminf_{n\to\infty} \|x_n\| \geq \|x_0\|\,.$$

6. Let E and F be Banach spaces. Suppose $T : E \to F$ is a linear map such that
$$\phi \circ T \in E^*, \quad \forall\, \phi \in F^*\,.$$
Show that T is bounded.

7. (a) State Liouville's theorem.
   (b) Suppose f is entire and such that there exists a constant $C > 0$ and $p \in \mathbb{N}$ such that $|f(z)| \leq C|z|^p$ for every z with $|z| \geq 1$. Prove, using Liouville's theorem, that f is a polynomial of degree at most p.

8. (a) Prove using complex integration that for any $a \in \mathbb{R}$, $\int_{\mathbb{R}} e^{-(x+ia)^2}\, dx = \int_{\mathbb{R}} e^{-x^2}\, dx$. Write in detail any limiting process involved.
   (b) Let $\sigma > 0$ fixed. Use part (a) to show that
$$\int_{\mathbb{R}} e^{-ix\xi} e^{-\frac{x^2}{2\sigma^2}}\, dx = \sigma\sqrt{2\pi}\, e^{-\frac{\sigma^2\xi^2}{2}}, \qquad \xi \in \mathbb{R}.$$
   You may use without proof that $\int_{\mathbb{R}} e^{-x^2}\, dx = \sqrt{\pi}$.