Winter 2019 - Analysis prelim - Friday, January 18
University of California Santa Cruz

1. Let K be a compact metric space and let A be a subset of $\mathcal{C}(K)$ (i.e., the space of all continuous real functions defined on K with the metric induced by the sup norm). Prove that A is compact if and only if A is closed, bounded and equicontinuous.

2. Suppose that $U \subset \mathbb{R}^n$ is an open set, $K \subset U$ and K is compact. Prove that there is an open set V whose closure is compact such that
$$K \subset V \subset \overline{V} \subset U\,.$$
Please state clearly the statements of the theorems you are using and verify the assumptions needed to apply the theorems.

3. Let f and g be absolutely continuous functions on $[0, 1]$. Show that their product is also absolutely continuous.

4. Let (X, Σ, μ) be a finite measure space (i.e., $\mu(X) < \infty$), $\{E_k\}_{k=1}^n$ a collection of measurable sets, and $\{c_k\}_{k=1}^n$ a collection of real numbers. For $E \in \Sigma$, define
$$\nu(E) = \sum_{k=1}^{n} c_k \ \mu(E \cap E_k).$$
Show that $\nu \ll \mu$ and find the Radon-Nikodym derivative $\frac{d\nu}{d\mu}$.

5. Let X be a Banach space, and let $B = \{x \in X : \|x\| \leq 1\}$ be the closed unit ball. Show that B is closed in the weak-topology. Give an example of a Banach space such that the unit sphere $S = \{x \in X : \|x\| = 1\}$ is not closed in the weak-topology.

6. Show that the spectrum of a bounded linear operator A on a complex Banach space X is a non-empty compact subset of $\mathbb{C}$.

7. (a) State Rouché's theorem.
   (b) Let $p_n(z) = z^2 - 2\left(\frac{z}{3} + \frac{z^2}{9} + \frac{z^3}{27} + \ldots \frac{z^n}{3^n}\right)$, and let Z_n the number of zeros of $p_n(z)$, taking multiplicities into account, in the disk $\{z \in \mathbb{C}, |z - 1/2| < 1\}$. Determine the limit $\lim_{n\to\infty} Z_n$.

8. Let $p \geq 1$ be a fixed integer. Using residue theory, compute the integral
$$\int_{\mathbb{R}} \frac{1}{1 + x^{2p}}\ dx.$$
Hint: use the sector $S_R = \left\{re^{i\theta}, r \in (0, R), \theta \in (0, \frac{\pi}{p})\right\}$. Justify any limiting processes involved.