Geometry–Topology Prelim, Fall 2012

1. For which values of the constant c is the level set $x^2 + y^2 - zw = c$ a submanifold of $\mathbb{R}^4$?

2. Consider the vector field $v = \sum_i x_i \dfrac{\partial}{\partial x_i}$ on $\mathbb{R}^n$.

(a) Find the flow of v.
(b) Prove that the only one-form on $\mathbb{R}^n$ invariant under the flow of v is the trivial one-form $\alpha = 0$.

3. Recall that the fundamental group $\pi_1(X, x_0)$ is the set of homotopy classes of base point preserving maps $(S^1, s_0) \to (X, x_0)$. Let $[S^1, X]$ be the set of free homotopy classes of maps without conditions on base points. Then there exists a map $\Phi\colon \pi_1(X, x_0) \to [S^1, X]$ obtained by ignoring base points.

(a) Show that Φ is onto if X is path connected.
(b) $\Phi([f]) = \Phi([g])$ if and only if $[f]$ and $[g]$ are conjugate in $\pi_1(X, x_0)$.
(c) Let X be the one point union of two real projective planes $X = \mathbb{R}P^2 \vee \mathbb{R}P^2$ joined at a point x_0. Compute $\pi_1(X, x_0)$. (You can use the fact that $\pi_1(\mathbb{R}P^2) = \mathbb{Z}_2$.)
(d) Determine the set of free homotopy classes of maps $[S^1, X]$.

4. Let $f\colon S^n \to S^n$ be a continuous map. Assume that f is fixed point free, i.e., $f(x) \neq x$ for all $x \in S^n$.

(a) Show that f is homotopic to the antipodal map.
(b) Assume that n is even. Show that there exists a point such that $f(x) = -x$.
(c) Show that for any odd n there exists a continuous map $f\colon S^n \to S^n$ such that $f(x) \neq x$ and $f(x) \neq -x$ for all x. (Hence the condition that n is even is essential in (b).)

5.

(a) Find the Gaussian and mean curvature of the surface $z = x\,y$ with the Riemannian structure induced by the standard Euclidean structure on $\mathbb{R}^3$.
(b) If M is a surface in $\mathbb{R}^3$, then a Euclidean isometry φ of $\mathbb{R}^3$ such that $\varphi(M) = M$ is called a *Euclidean symmetry of M*. Show that every Euclidean symmetry of the surface $z = x\,y$ is an orthogonal transformation, i.e., a linear transformation preserving the Euclidean inner product on $\mathbb{R}^3$.
(c) Determine which orthogonal transformations are Euclidean symmetries of the surface $z = x\,y$.