

**Geometry and Topology Preliminary Exam, Fall 2014**

[1] Which of the following manifolds are diffeomorphic and which are not:

- (a)  $\mathbb{R}P^2$ ,  $\mathbb{C}P^1$ , and  $S^2$ .
- (b)  $\mathbb{R}P^3$ ,  $S^3$ ,  $SO(3)$ ,  $SU(2)$ , and the unit tangent bundle to  $S^2$ .

Justify your conclusions.

[2] Consider the following two vector fields  $v, w$  on the plane

$$v = x \frac{\partial}{\partial y} + y \frac{\partial}{\partial x}, \quad w = x^2 \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y}.$$

- (a) Are these vector fields complete?
- (b) Find the flow of  $v$ .
- (c) Find the bracket  $[v, w]$ .

[3] Consider the map of wedge product  $\phi : \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \Lambda^2 \mathbb{R}^3$  which sends  $v, w$  to their wedge product  $v \wedge w$ .

- (i) What are the critical points of  $\phi$ ?
- (ii) What are the critical values of  $\phi$ ?
- (iii) What is the dimension of the image of  $\phi$ ?
- (iv) What is the image of  $\phi$ ?

[4] Let  $M = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$  be the cylinder of unit radius.

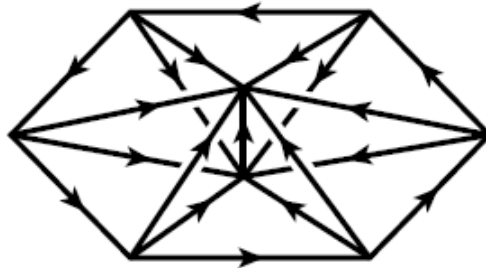
(a) Show that the geodesics on  $M$  are the helices, that is, curves which cut each generator (= each vertical line) at the same angle (or have a constant angle with the  $z$ -axis), the generators themselves, and the circles of intersection with planes  $z = \text{constant}$ .

(b) How many geodesics connect two given points  $p, q$  on  $M$ ?

(c) Show that a geodesic starting at a point  $(x, y, z)$  in  $M$  does not minimize arc length after it passes through the antipodal line  $\{(-x, -y, t) \mid t \in \mathbb{R}\}$ .

[5] On a closed orientable surface  $\Sigma_{g+h}$  of genus  $g + h$  with  $g, h \geq 0$ , let  $C$  be a loop that separates  $\Sigma_{g+h}$  into two compact surfaces  $\Sigma'_g = \Sigma_g - \{\text{open disc}\}$  and  $\Sigma'_h = \Sigma_h - \{\text{open disc}\}$  of genus  $g$  and  $h$ , respectively. Show that  $\Sigma'_g$  does not retract onto its boundary  $C$ , and hence  $\Sigma_{g+h}$  does not retract onto  $C$ .

[6] Construct a 3-dimensional  $\Delta$ -complex  $X$  from four oriented tetrahedra  $T_1, T_2, T_3, T_4$  by the following two steps. (The picture below uses six tetrahedra. Here we use **four** for simplicity.) First arrange the tetrahedra in a cyclic pattern so that all four tetrahedra share the same single edge, and each  $T_i$  shares a common vertical face with its two neighbors  $T_{i-1}$  and  $T_{i+1}$ , subscripts taken mod 4. Then identify the bottom face of  $T_i$  with the top face of  $T_{i+1}$  by orientation preserving homeomorphism for each  $i$ . Show that the homology of  $X$  in dimensions 0, 1, 2, 3 are  $\mathbb{Z}, \mathbb{Z}_4, 0, \mathbb{Z}$ .



[7] Show that if  $M$  is a compact smooth orientable surface in  $\mathbb{R}^3$  that is not diffeomorphic to a sphere, then there is a point  $p$  in  $M$  at which the Gaussian curvature is negative.