Geometry and Topology Preliminary Examination **Fall 2015**

Instructions: Answer seven of the following eight problems. Indicate clearly which problems you want graded—we won't pick your best ones for you.

1.	**2.**	**3.**	**4.**	**5.**	**6.**	**7.**	**8.**

1. Prove or provide a counter-example: Any integral manifold of a smooth vector field on a manifold M is an embedded submanifold of M.

2. Let M be an n-dimensional Riemannian manifold. Define the *unit tangent bundle* $UT(M)$ of M by
$$UT(M) := \coprod_{m\in M} \{v_m \in T_mM : ||v_m||_m = 1\}.$$
Show that $UT(M)$ is a $2n-1$ dimensional embedded submanifold of TM.

3. (a) Let X be a smooth vector field on a smooth manifold M. Prove or provide a counter-example: If β is a closed form on M, then the Lie derivative $\mathcal{L}_X\beta$ of β is exact.
   (b) Find the Lie derivative of the two-form $\beta(x, y) = \left(x^k + y^k\right) dx \wedge dy$ on $\mathbb{R}^2$ with respect to the vector field $X(x, y) = x^\ell \partial_y$ for $k, \ell \in \mathbb{N}$.

4. Show that if A is a retract of X, then the map $H_n(A) \to H_n(X)$ induced by the inclusion $A \subset X$ is injective.

5. Suppose $F : M \to N$ and $G : N \to P$ are smooth maps, and G is transverse to an embedded submanifold $X \subseteq P$. Show that F is transverse to $G^{-1}(X)$ if and only if $G \circ F$ is transverse to X.

6. A polynomial $p : \mathbb{C} \to \mathbb{C}$ can be extended to a continuous map $\hat{p} : S^2 \to S^2$ of one-point compactifications. Show that the degree of $\hat{p}$ equals the degree of p as a polynomial.

7. A connection ∇ on the tangent bundle TM of a smooth manifold M induces a connection ∇^* on the cotangent bundle T^*M by requiring
$$X(\iota_Y \alpha) = \iota_Y \nabla^* X\alpha + \iota_{\nabla_X Y}\alpha$$
for all smooth vector fields X and Y, and all one-forms α. Show that the condition that ∇ is torsion free is equivalent to the condition that for any one-form α and smooth vector fields X and Y we have
$$d\alpha(X, Y) = \nabla^*_X\alpha(Y) - \nabla^*_Y\alpha(X).$$

8. Prove or provide a counter-example: If a Lie group G acts transitively on a smooth manifold M, then any G-equivariant vector field on M is complete.