GEOMETRY AND TOPOLOGY
PRELIMINARY EXAMINATION JUNE 2008

1. Prove that there is no retraction

$$r : B^{n+1} \to S^n,$$

where $B^{n+1} = \{x \in R^{n+1} : ||x|| \leq 1\}$ and $S^n = \{x \in R^{n+1} : ||x|| = 1\}$.

2. Let V and W be finite-dimensional real vector spaces. Prove that there is a canonical isomorphism between

$$V^* \bigotimes W \simeq \mathrm{Hom}(V, W),$$

where $\mathrm{Hom}(V, W)$ is the space of all linear map from V to W.

3. Let $(M,\ g)$ be a Riemannian manifold and $p \in M$. Prove that, given $v \in T_pM$ and the geodesic $\gamma_v(t) = exp_p(tv)$, we have
(1) $dexp_p(tv)v = \gamma_v'(t)$
(2) $< dexp_p(tv)\xi, \gamma_v'(t) >=< \xi, v >$.
Consequently, we have $||dexp_p(tv)v|| = ||v||$ and $dexp_p(tv)\xi \perp \gamma_v'(t)$ if $\xi \perp v$.

4.
a. Let M and N be smooth manifolds and let $F : M \to R^n$ and $G : N \to R^n$ be smooth maps with disjoint images. Assume that $\dim(M) + \dim(N) < n - 1$. Prove that there is a point in R^n which does not lie on a straight line intersecting both $F(M)$ and $G(N)$.
b. Does the above assertion 4(a) remain correct if one of the maps F, G is only continuous?

5. Let $(M^{2n},\ \omega)$ be a smooth, connected, closed manifold with a symplectic form ω. Show that the homology groups $H_{2k}(M, Z) \neq 0$ for all $k = 0, 1, \cdots n$.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

6.

a. Let p be a point in $S^1 \times S^1$. Compute all the homology groups $H_*(S^1 \times S^1 \setminus \{p\}, Z)$.

b. A surface Σ_2 of genus 2 may be obtained by gluing two copies of $S^1 \times S^1 \setminus \{p\}$ as follows:

Use the Mayer-Vietoris Theorem to compute all the homology groups $H_*(\Sigma_2, Z)$.

7. Prove or disprove: The Lie group $SU(2)$ contains the two-torus T^2 as a Lie subgroup.

8. On R^4, construct two vector fields X and Y with the property that $X, Y, [X, Y]$ and $[X, [X, Y]]$ span all of R^4 at every point. Write out the vector fields in terms of standard coordinates x, y, z, w and their associated coordinate vector fields.